DOCUMENT RESUME

ED 177 019 SE 029 186

AUTHOR	Weaver, J. Fred; And Others
TITLE	Calculator Explorations with Seventh Grade Students: Some Calculator-Inspired Instructional Materials, Observations, and Investigations. Technical Report No. 497.
INSTITUTION	Wisconsin Univ., Madison. Research and Development Center for Individualized Schooling.
SPONS AGENCY	National Inst. of Education (DHEW), Washington, D.C.
PUB DATE	Jul 79
GRANT	OB-NIE-G-78-0217
NOTE	248p.; Report from the Project on Studies in Mathematics; Best copy available
EDRS PRICE	MF01/PC10 Plus Postage.
DESCRIPTORS	Calculation; Experimental Curriculum; Grade 7; *Instructional Materials; *Junior High School Students; *Mathematics Curriculum; *Mathematics Instruction; Middle Schools; *Research; Secondary Education; *Secondary School Mathematics; Worksheets
IDENTIFIERS	*Calculators

ABSTRACT

This report brings together in a single source observations and materials of instructional importance in calculator explorations with seventh-grade students from 1976 through 1978. Students used algebraic-logic calculators with no operational hierarchies. At the end of the school year, students were introduced to reverse Polish notation-based calculators. Student performance with number sentences and with calculator notation and tasks is described. Illustrative materials are included. (Author/MK)

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Technical Report No. 497

CALCULATOR EXPLORATIONS WITH SEVENTH GRADE STUDENTS: SOME CALCULATOR-INSPIRED INSTRUCTIONAL MATERIALS, OBSERVATIONS, AND INVESTIGATIONS

BEST COPY AVAILABLE

by

J. Fred Weaver, Glendon W. Blume, and Charles E. Mitchell

Report from the Project on
Studies in Mathematics

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Wisconsin Research and Development Center
for Individualized Schooling
The University of Wisconsin
Madison, Wisconsin

July 1979

SE 029 186

Published by the Wisconsin Research and Development Center for Individualized Schooling. The project presented or reported herein was performed pursuant to a grant from the National Institute of Education, Department of Health, Education, and Welfare. However, the opinions expressed herein do not necessarily reflect the position or policy of the National Institute of Education, and no official endorsement by the National Institute of Education should be inferred.

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- providing assistance to educators which helps transfer the outcomes of research and development to improved practice in local schools and teacher education institutions

The Wisconsin Research and Development Center is supported with funds from the National Institute of Education and the University of Wisconsin.

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Acknowledgments

Sincere appreciation is expressed to Leonard Rush (Principal), to Ms. Jo Marshall (Learning Coordinator), and to Ms. Judi Didcoct and Jack Novick (teachers), and their students, for their interest and cooperation which made possible the calculator-influenced explorations.

Acknowledgment is also made of the various ways in which Mr. Blume and Mr. Mitchell, Ph.D. candidates in mathematics education, with the various phases of the explorations.

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Abstract

This report brings together in a single source observations and materials of instructional importance in calculator explorations with seventh grade students from 1976 through 1978. Students used algebraic-logic calculators with no operational hierarchies. At the end of the school year, students were introduced to reverse Polish notation-based calculators.

INTRODUCTION

by

J. Fred Weaver

These observations and materials accrue from work done during the 1976-77 and 1977-78 school years with groups of seventh grade students at the Jefferson Middle School, Madison, Wisconsin.

1976-1977 School Year

A group of approximately 30 students, drawn from two classes, provided the sample for the work, meeting for 30 minutes once each week during most of the year. Teachers were at liberty to use calculators with their students at other times, in whatever way and to whatever extent they wished.

During most of the year, students used algebraic-logic calculators with no operational hierarchies: first, calculators whose keyboard is depicted in Figure 1.1 (non-nesting parentheses, no "memory"); later, those whose keyboard is depicted in Figure 1.2 (no parentheses, but a store/recall "memory" in which it also was possible to operate, (add, subtract, multiply, or divide) within the memory.

During a few weeks at the end of the school year, students were introduced to reverse Polish notation-based (RPN) calculators using those whose keyboard is represented in Figure 1.3. This experience led to a subsequent keyboard modification in labeling, as shown in Figure 1.4.

Chapters 1, 2, and 3 of this report pertain particularly to the work of the 1976-77 school year.

1977-1978 School Year

A class of approximately 40 students, tracked to study first-year algebra in eighth rather than ninth grade, provided the sample for the

16

work of this year, meeting for 40 minutes once each week throughout the year. The teacher was at liberty to use calculators with the students at other times, to whatever extent and in whatever way desired.

Instruction involved only RPN calculators of the kind whose keyboard is illustrated in Figure 1.5, but modified in accord with Figure 1.6. A 3-level stack was incorporated in this model. Most other RPN calculators have a 4-level stack.

Considerable use was made of a programmable demonstration calculator whose keyboard is represented in Figure 1.7.

Flowcharts 1, 2, 3, 3A, 4, 5, and 5A were used to advantage in introducing students to basic RPN operational procedures. Flowchart 6 is included here to show the increasing complexity of such representations if Flowchart 5 or 5A is not used.

Chapters 5 through 9 of this report pertain to, or accrue from, the work of the 1977-78 school year.

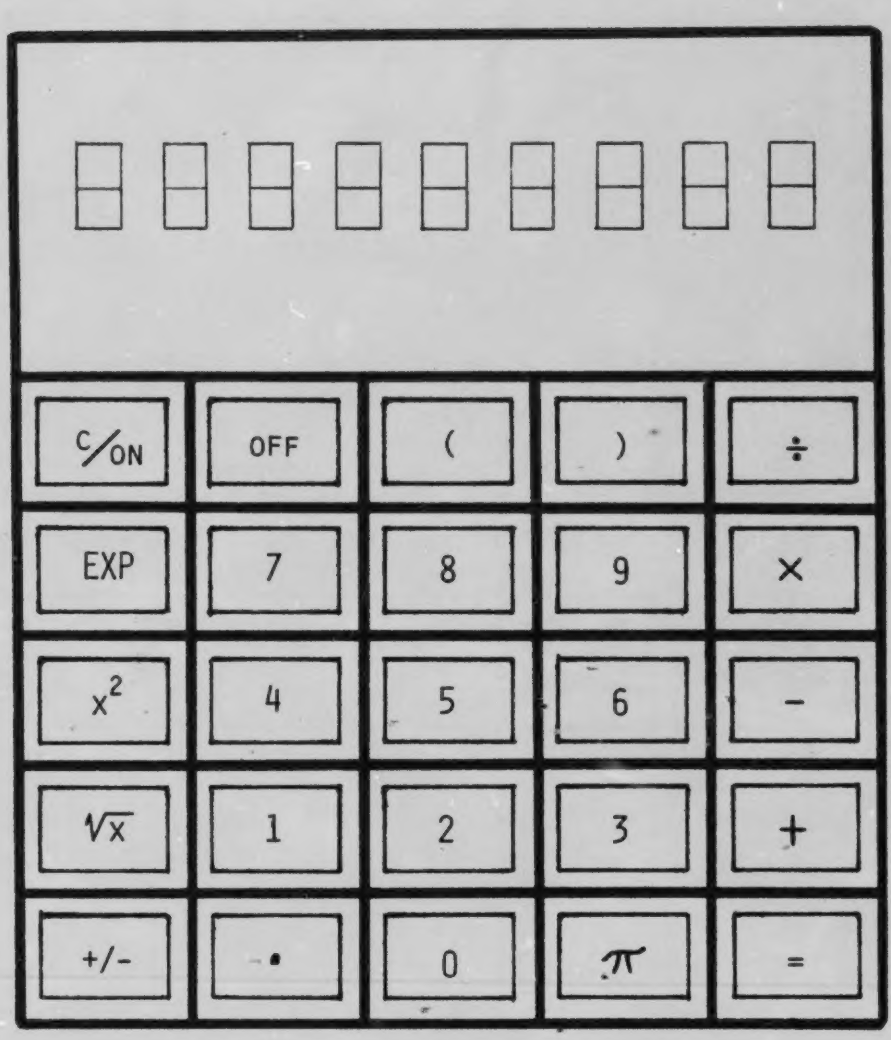

C/ON
OFF
(
)
÷
EXP
7
8
9
×
x²
4
5
6
−
√x
1
2
3
+
+/-
•
0
π
=

Figure 1.1. Facsimile of Litronix 2260 Calculator Keyboard

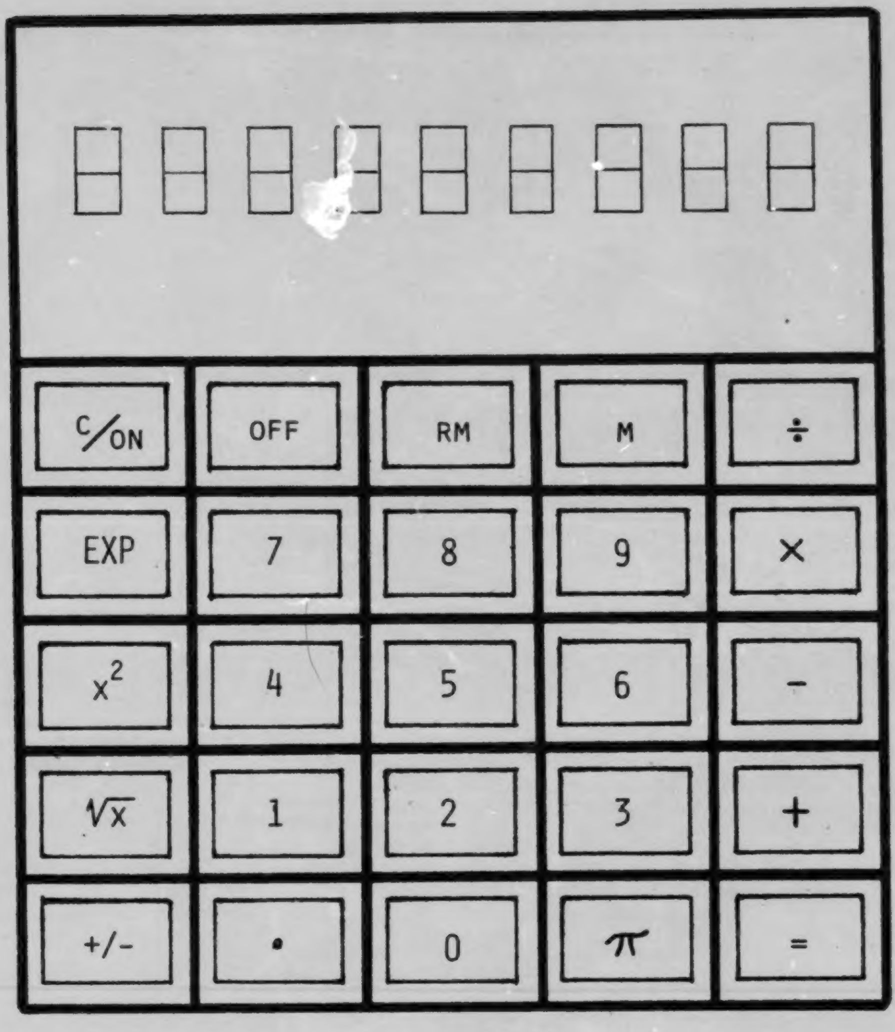

C/ON
OFF
RM
M
÷
EXP
7
8
9
×
x²
4
5
6
−
√x
1
2
3
+
+/-
•
0
π
=

Figure 1.2. Facsimile of Litronix 2250 Calculator Keyboard

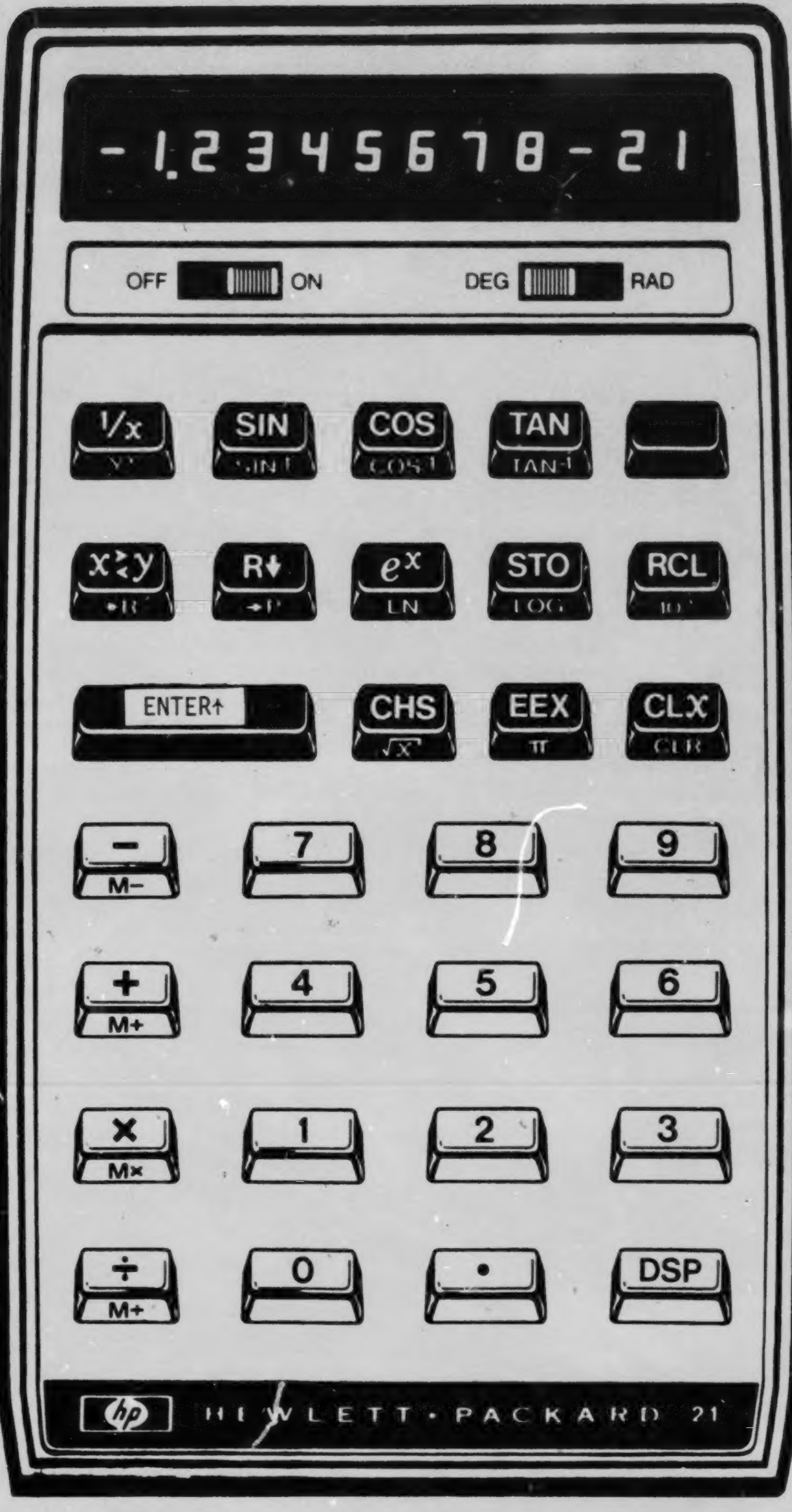

-1.2345678-21
OFF
ON
DEG
RAD
1/x
SIN
COS
TAN
x⇄y
R↓
e^x
LN
STO
LOG
RCL
ENTER↑
CHS
√x
EEX
π
CLX
CLR
−
M−
7
8
9
+
M+
4
5
6
×
M×
1
2
3
÷
M÷
0
•
DSP
hp
HEWLETT · PACKARD 21

Figure 1.3. Facsimile of Hewlett-Packard HP-21 Calculator Keyboard

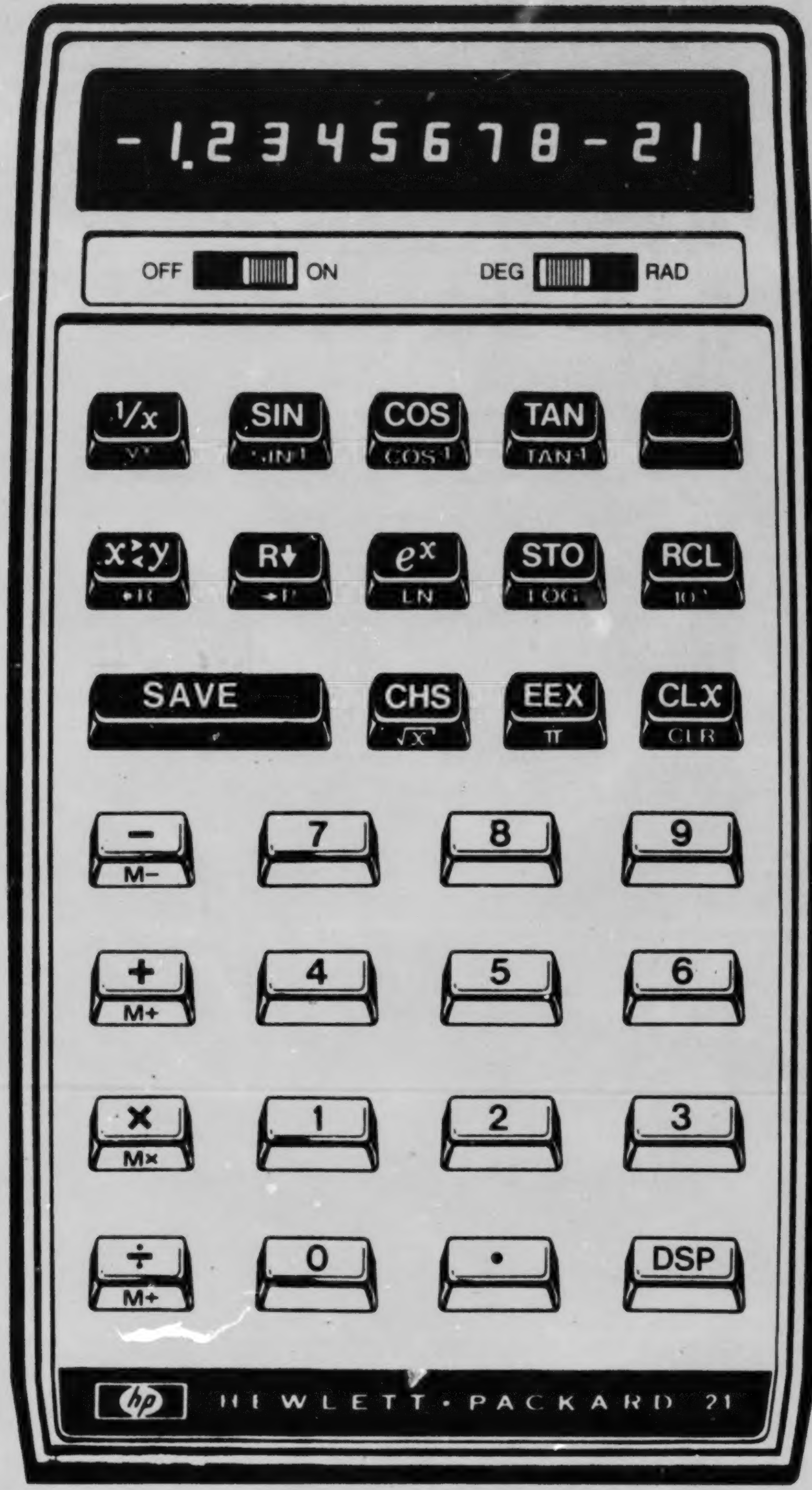

-1.2345678-21
OFF
ON
DEG
RAD
1/x
SIN
COS
TAN
y^x
SIN-1
COS-1
TAN-1
x⇄y
R↓
e^x
STO
RCL
→R
→P
LN
LOG
10^x
SAVE
CHS
EEX
CLX
√x
π
CLR
−
M−
7
8
9
+
M+
4
5
6
×
M×
1
2
3
÷
M÷
0
•
DSP
hp
HEWLETT • PACKARD 21

Figure 1.4. Facsimile of Modified HP-21 Keyboard.

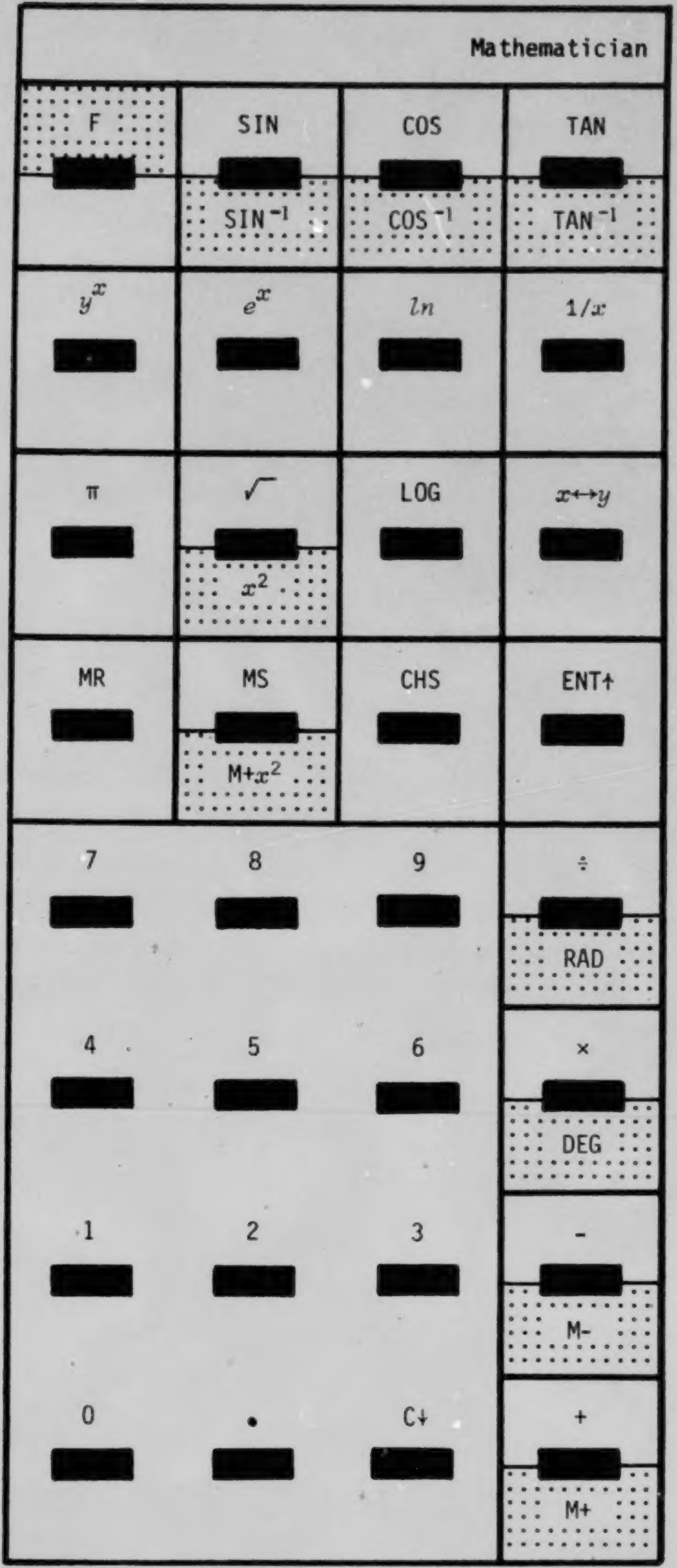

Mathematician
F
SIN
SIN⁻¹
COS
COS⁻¹
TAN
TAN⁻¹
y^x
e^x
ln
$1/x$
π
√
x^2
LOG
$x\leftrightarrow y$
MR
MS
$M+x^2$
CHS
ENT↑
7
8
9
÷
RAD
4
5
6
×
DEG
1
2
3
-
M-
0
•
C↓
+
M+

Figure 1.5. Facsimile of National Semiconductor 4510 Calculator Keyboard

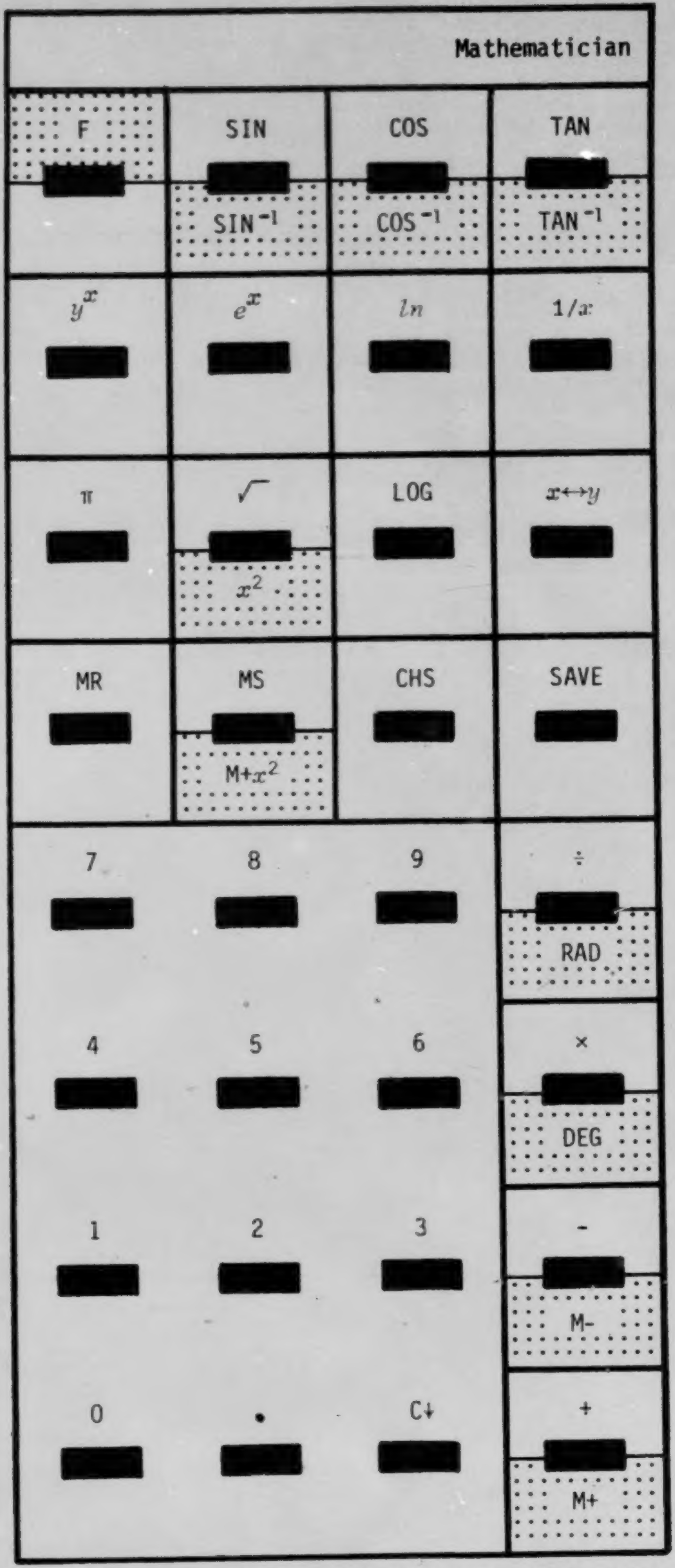

Mathematician
F
SIN
COS
TAN
SIN^{-1}
COS^{-1}
TAN^{-1}
y^x
e^x
ln
$1/x$
π
√
LOG
$x \leftrightarrow y$
x^2
MR
MS
CHS
SAVE
M+x^2
7
8
9
÷
RAD
4
5
6
×
DEG
1
2
3
-
M-
0
•
C↓
+
M+

Figure 1.6. Facsimile of Modified NS 4510 Keyboard

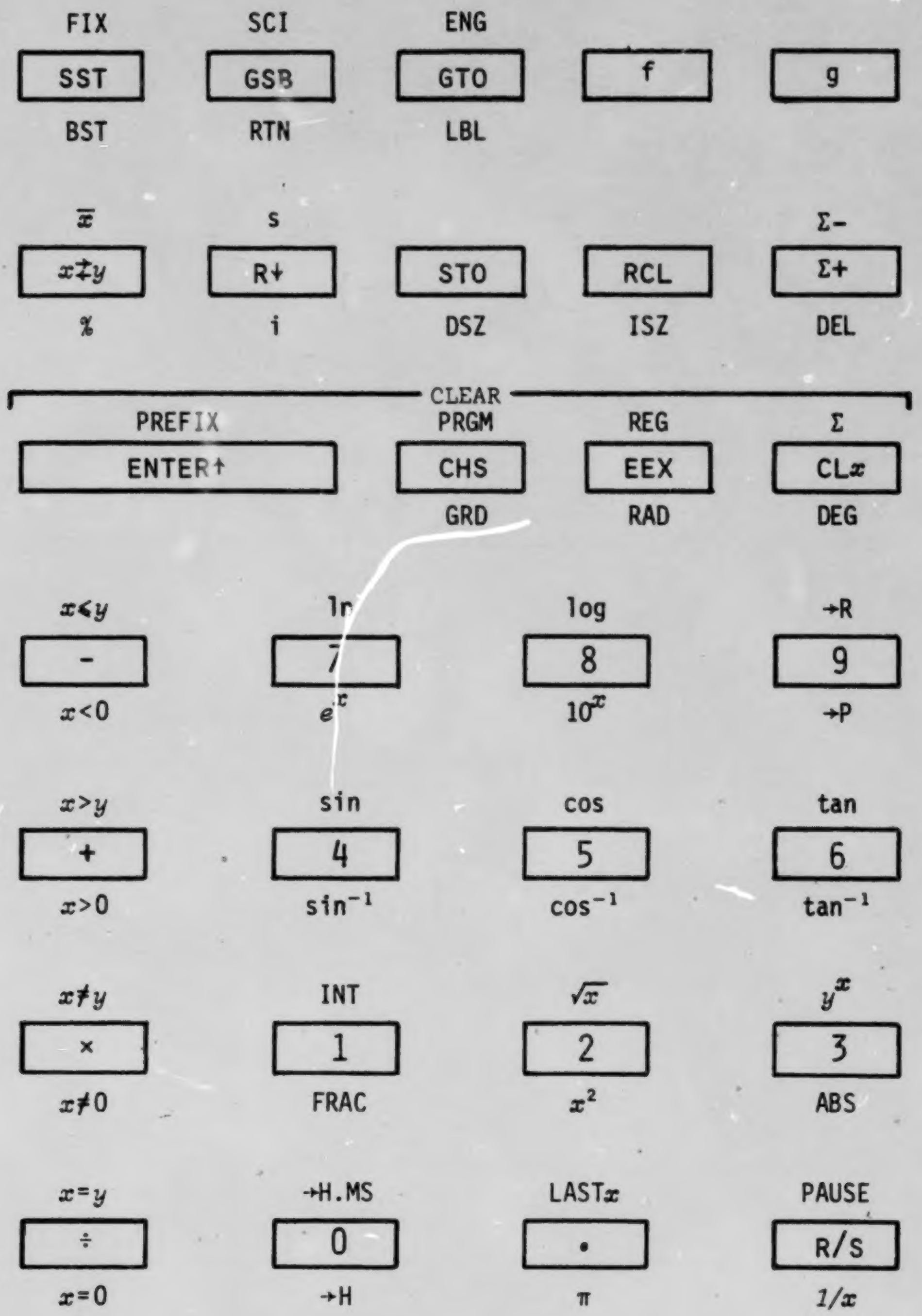
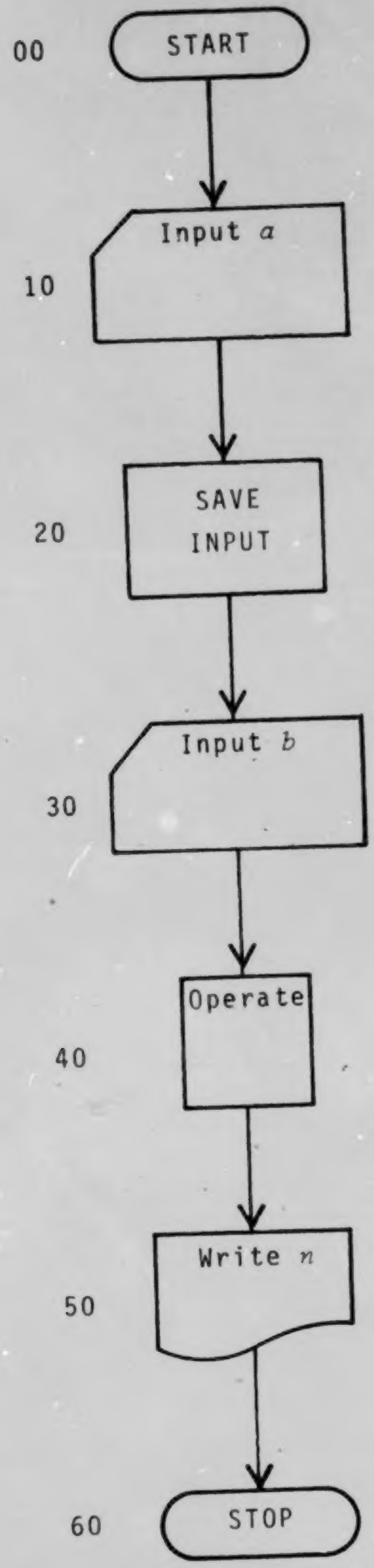

FIX
SST
BST
SCI
GSB
RTN
ENG
GTO
LBL
f
g
x̄
x⇄y
%
s
R↓
i
STO
DSZ
RCL
ISZ
Σ-
Σ+
DEL
CLEAR
PREFIX
ENTER↑
PRGM
CHS
GRD
REG
EEX
RAD
Σ
CLx
DEG
x≤y
-
x<0
ln
7
e^x
log
8
10^x
→R
9
→P
x>y
+
x>0
sin
4
sin⁻¹
cos
5
cos⁻¹
tan
6
tan⁻¹
x≠y
×
x≠0
INT
1
FRAC
√x
2
x²
y^x
3
ABS
x=y
÷
x=0
→H.MS
0
→H
LASTx
•
π
PAUSE
R/S
1/x

Figure 1.7. Facsimile of Hewlett-Packard HP-29C Calculator Keyboard used in conjunction with the EduCALC 29CGD.

$a * b = n$

$$\begin{array}{r} a \\ \underline{* \, b} \\ n \end{array}$$

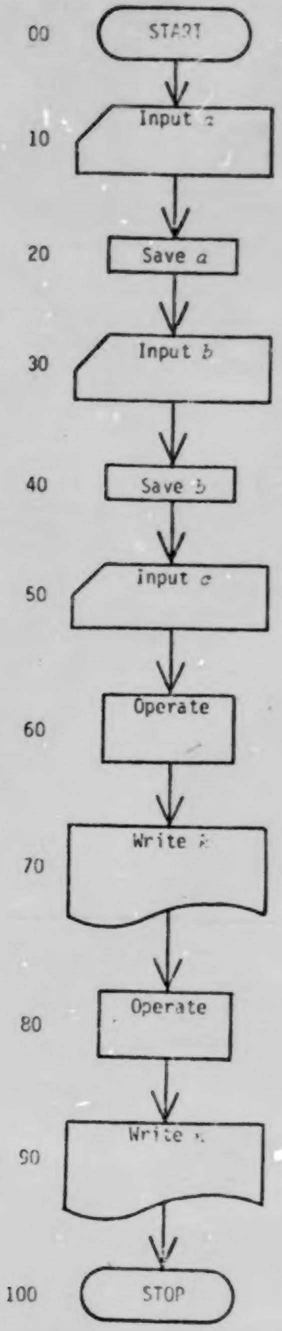
Flowchart #1


00
START
10
Input a
20
SAVE INPUT
30
Input b
40
Operate
50
Write n
60
STOP

Flowchart #2

$a * (b \# c) = n$

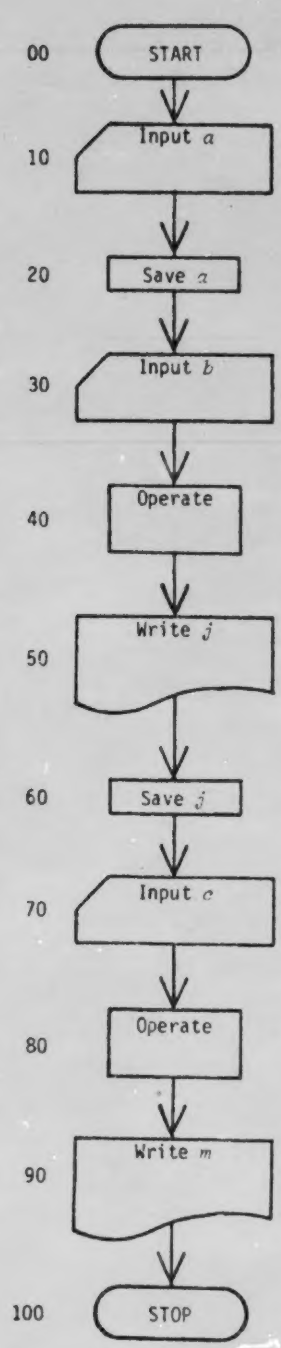
$a * k = n$


00
START
10
Input a
20
Save a
30
Input b
40
Save b
50
input c
60
Operate
70
Write k
80
Operate
90
Write n
100
STOP

Flowchart #3

$(a * b) \# c = m$

$j \# c = m$

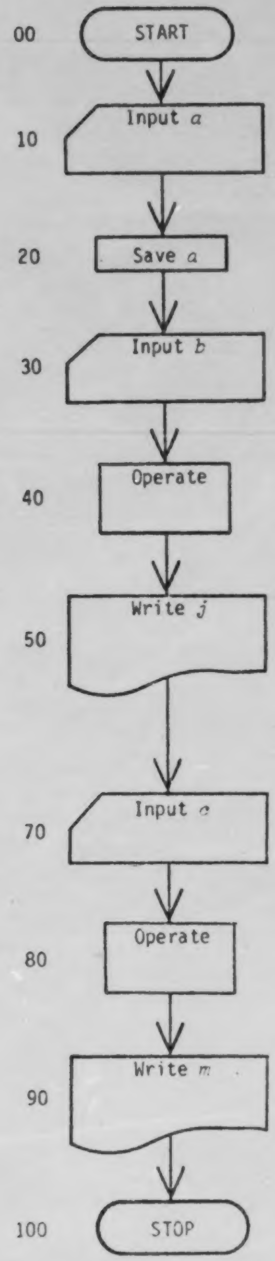

00
START
10
Input a
20
Save a
30
Input b
40
Operate
50
Write j
60
Save j
70
Input c
80
Operate
90
Write m
100
STOP

Flowchart #3A

(shorter form)

$(a * b) \# c = m$

$j \# c = m$

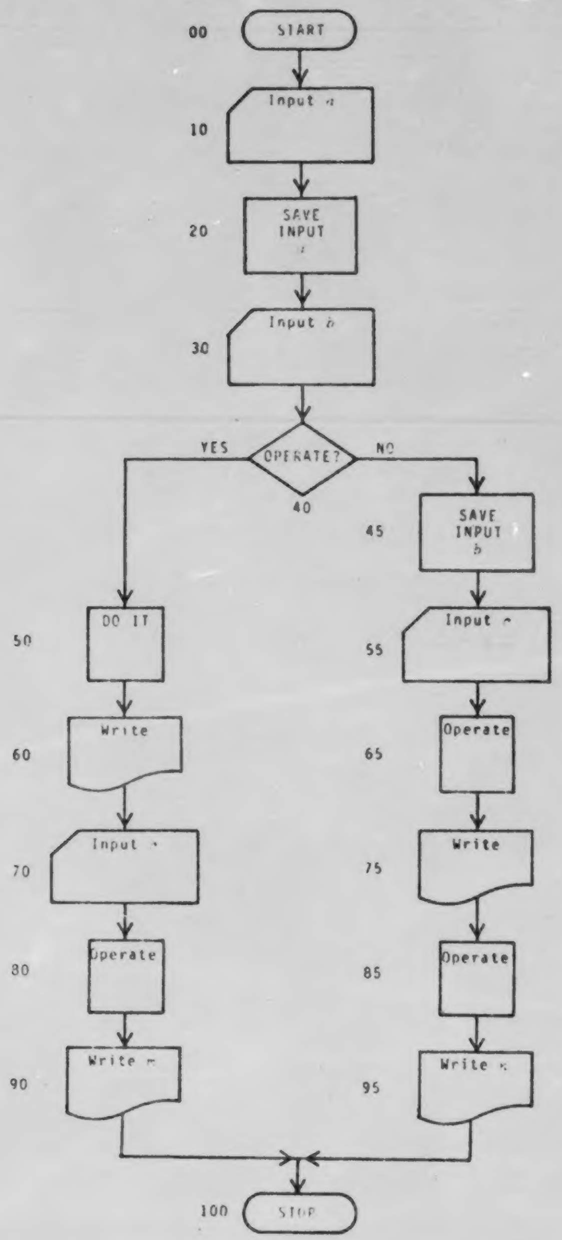

00
START
10
Input *a*
20
Save *a*
30
Input *b*
40
Operate
50
Write *j*
70
Input *c*
80
Operate
90
Write *m*
100
STOP

Flowchart #4

*(a * b) # c = m*

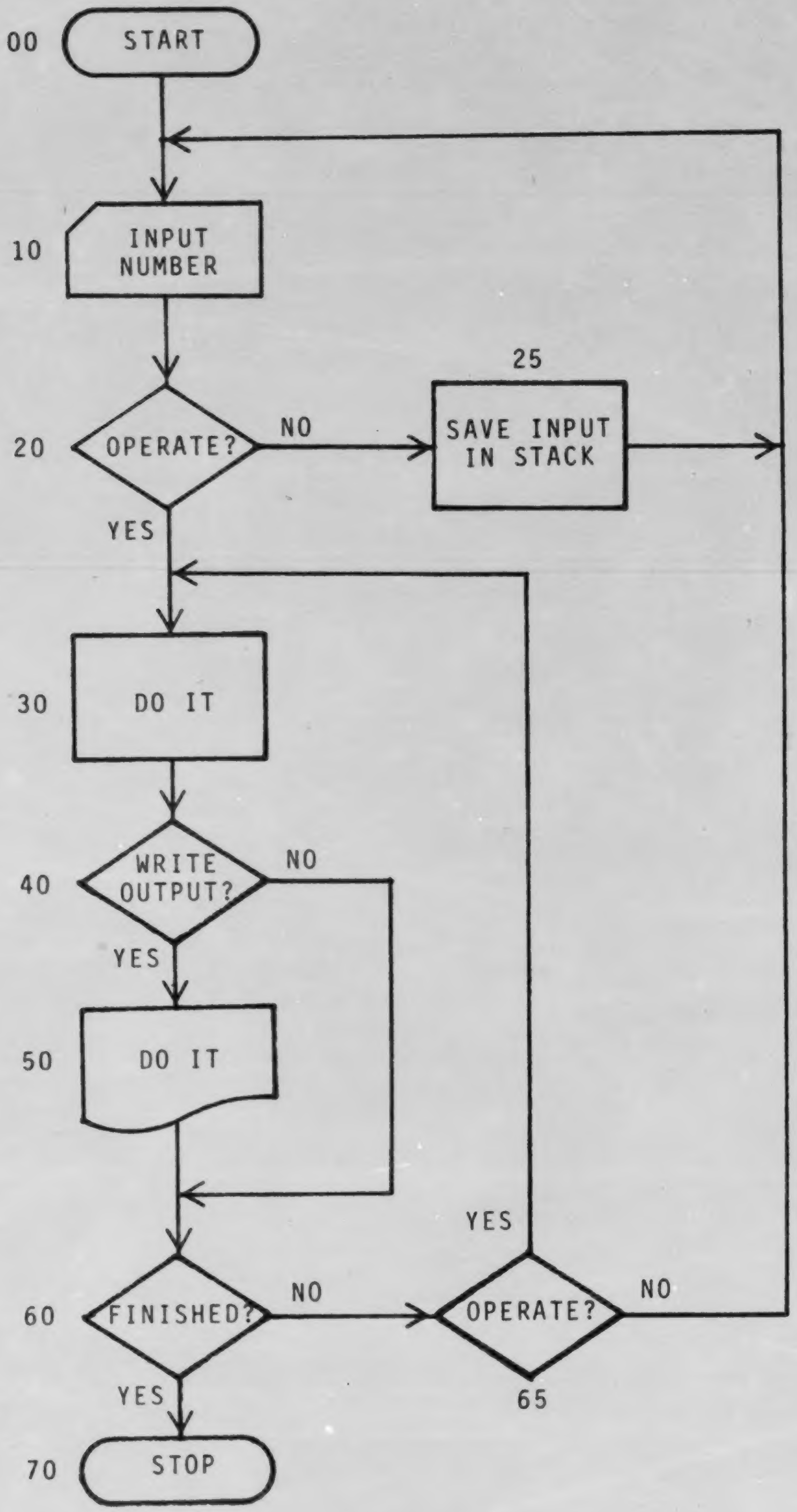
or

*a * (b # c) = n*


00
START
10
Input a
20
SAVE
INPUT
a
30
Input b
YES
OPERATE?
NO
40
45
SAVE
INPUT
b
50
DO IT
55
Input c
60
Write
65
Operate
70
Input c
75
Write
80
Operate
85
Operate
90
Write m
95
Write n
100
STOP

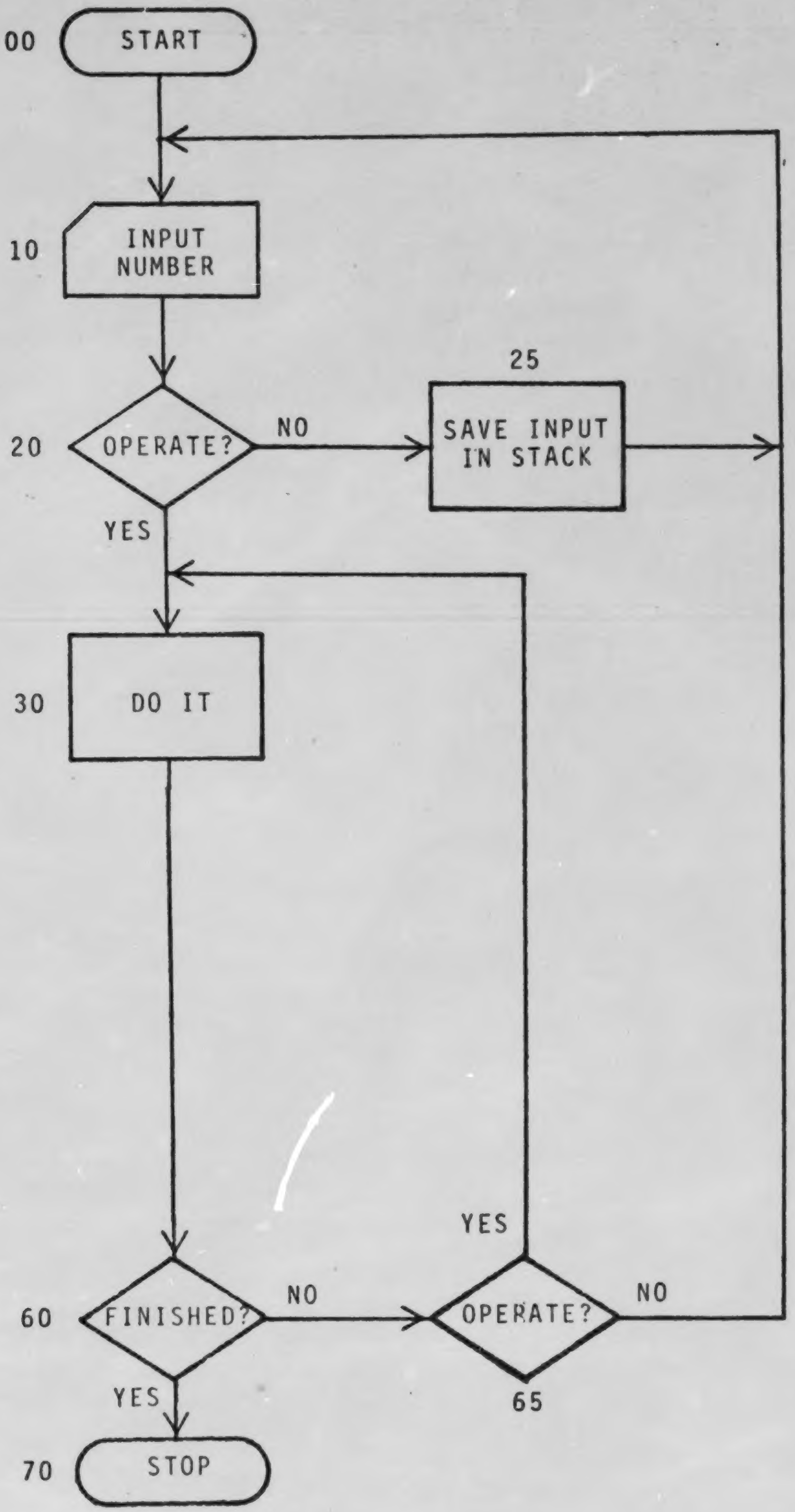
Flowchart #5

Learning to Use the Calculator


00
START
10
INPUT NUMBER
20
OPERATE?
NO
25
SAVE INPUT IN STACK
YES
30
DO IT
40
WRITE OUTPUT?
NO
YES
50
DO IT
60
FINISHED?
NO
YES
OPERATE?
YES
NO
65
70
STOP

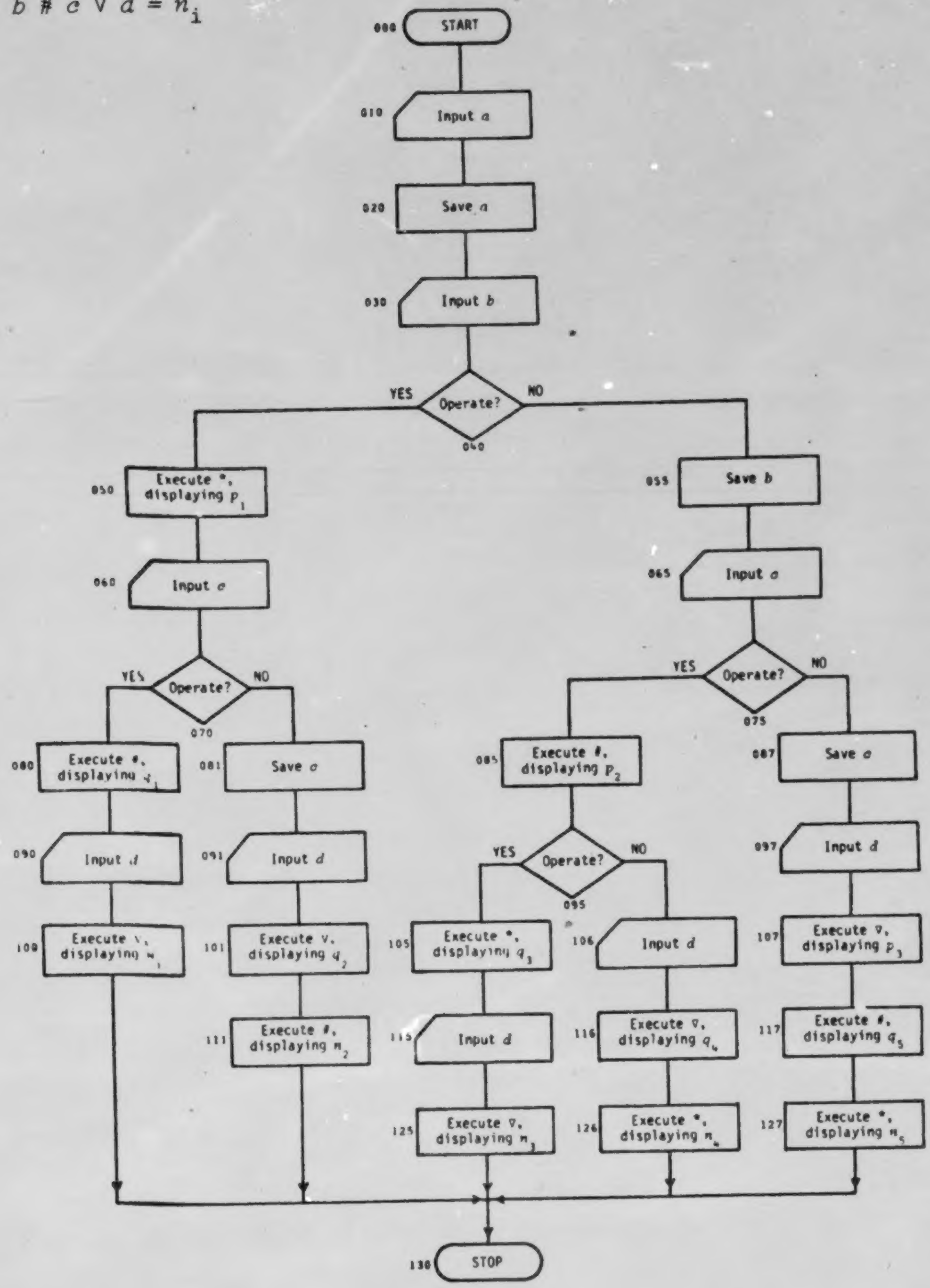
Flowchart #5A

Learning to Use the Calculator


00
START
10
INPUT NUMBER
20
OPERATE?
NO
25
SAVE INPUT IN STACK
YES
30
DO IT
YES
60
FINISHED?
NO
OPERATE?
NO
65
YES
70
STOP

Flowchart #6

$a * b \# c \nabla d = n_i$


000
START
010
Input a
020
Save a
030
Input b
YES
Operate?
NO
040
050
Execute *, displaying p_1
055
Save b
060
Input c
065
Input c
YES
Operate?
NO
070
YES
Operate?
NO
075
080
Execute #, displaying q_1
081
Save c
085
Execute #, displaying p_2
087
Save c
090
Input d
091
Input d
YES
Operate?
NO
095
097
Input d
100
Execute ∇, displaying n_1
101
Execute ∇, displaying q_2
105
Execute *, displaying q_3
106
Input d
107
Execute ∇, displaying p_3
111
Execute #, displaying n_2
115
Input d
116
Execute ∇, displaying q_4
117
Execute #, displaying q_5
125
Execute ∇, displaying n_3
126
Execute *, displaying n_4
127
Execute *, displaying n_5
130
STOP

I

STUDENT PERFORMANCE WITH NUMBER SENTENCES INVOLVING PARENTHETICAL PHRASES

by

J. Fred Weaver

33

In school mathematics, students often encounter number phrases embedded within number sentences (see Table 1.1). Consider exemplars of phrases of the forms

$$(a * b) \# c \quad \text{and} \quad a * (b \# c)$$

and exemplars of sentences of the forms

$$(a * b) \# c = n \quad \text{and} \quad a * (b \# c) = s$$

where * and # represent any of the four binary operations (+ or - or x or ÷) and where a, b, c are non-negative integers (whole numbers), so that a complete phrase or n or s is also a whole number.

The Problem

Prior to intervention by explicit instruction, how well do seventh grade students distinguish between equivalent and nonequivalent phrases or sentences; (i.e., between those that do and those that do not "name the same number" or in students' more common language, "have the same value"), given a pair of phrases or sentences having the same numbers and operations in the same order, and that differ only in parentheses positions?

Findings from various investigations including a pilot study preceding the investigation reported here (see Kasanda, 1977) suggested that the answer to this question may depend upon:

1. the particular operation(s) specified in the phrases or sentences;

2. certain features of the format in which phrases or sentences are presented to students; specifically as summarized in Table 1.2;

   a. the use of one vs. two sentences in displaying phrases to be compared; and

   b. in relation to parentheses position within each phrase, the order is presented in which the phrases are to be compared.

Table 1.1

Principal Types of Number Phrases

Without parentheses	With parentheses	
	Left(L)	Right(R)
1W. $a + b + c$	1L. $(a + b) + c$	1R. $a + (b + c)$
2W. $a + b - c$	2L. $(a + b) - c$	2R. $a + (b - c)$
3W. $a + b \times c$	3L. $(a + b) \times c$	3R. $a + (b \times c)$
4W. $a + b \div c$	4L. $(a + b) \div c$	4R. $a + (b \div c)$
5W. $a - b + c$	5L. $(a - b) + c$	5R. $a - (b + c)$
6W. $a - b - c$	6L. $(a - b) - c$	6R. $a - (b - c)$
7W. $a - b \times c$	7L. $(a - b) \times c$	7R. $a - (b \times c)$
8W. $a - b \div c$	8L. $(a - b) \div c$	8R. $a - (b \div c)$
9W. $a \times b + c$	9L. $(a \times b) + c$	9R. $a \times (b + c)$
10W. $a \times b - c$	10L. $(a \times b) - c$	10R. $a \times (b - c)$
11W. $a \times b \times c$	11L. $(a \times b) \times c$	11R. $a \times (b \times c)$
12W. $a \times b \div c$	12L. $(a \times b) \div c$	12R. $a \times (b \div c)$
13W. $a \div b + c$	13L. $(a \div b) + c$	13R. $a \div (b + c)$
14W. $a \div b - c$	14L. $(a \div b) - c$	14R. $a \div (b - c)$
15W. $a \div b \times c$	15L. $(a \div b) \times c$	15R. $a \div (b \times c)$
16W. $a \div b \div c$	16L. $(a \div b) \div c$	16R. $a \div (b \div c)$

Table 1.2

Exemplar-pair Characteristics

(Other than Number and Operation)

Order of parentheses positions	Phrase format	
	One-sentence	Two-sentence
First left, then right	$(a \star b) \# c = a \star (b \# c)$	$(a \star b) \# c = n$ $a \star (b \# c) = s$
First right, then left	$a \star (b \# c) = (a \star b) \# c$	$a \star (b \# c) = n$ $(a \star b) \# c = n$

Table 1.3

Distribution of 16 Principal Item Types Across Forms of Equivalence Inventory C

Item phrase		Two-sentence format				One-sentence format			
		Form 1		Form 2		Form 3		Form 4	
		Parentheses L(eft) or R(ight)							
		L R	R L	L R	R L	LR	RL	LR	RL
		Item number							
1	a + b + c		3	19			3	19	
2	a + b - c	19			8	19			8
3	a + b × c		1	6			1	6	
4	a + b ÷ c	12			17	12			17
5	a - b + c	9			4	9			4
6	a - b - c		17	12			17	12	
7	a - b × c		15	2			15	2	
8	a - b ÷ c	4			9	4			9
9	a × b + c		10	1			10	1	
10	a × b - c	2			20	2			20
11	a × b × c	13			3	13			3
12	a × b ÷ c		8	13			8	13	
13	a ÷ b + c	6			14	6			14
14	a ÷ b - c		14	18			14	18	
15	a ÷ b × c	20			15	20			15
16	a ÷ b ÷ c		18	10			18	10	

Table 1.4

Distribution of Other Phrase Combinations
Across Forms of Equivalence Inventory C

Phrase combination			Two-sentence format				One-sentence format			
			Form 1		Form 2		Form 3		Form 4	
			Parentheses L(eft) or R(ight)							
			L R	R L	L R	R L	LR	RL	LR	RL
#	First phrase	Second phrase	Item number							
22	(a ÷ b) + c	c + (a ÷ b)	16				16			
33	a × (b - c)	(b - c) × a				11				11
44	(a + b) ÷ c	c ÷ (a + b)			5				5	
55	a - (b × c)	(b × c) - a		5				5		
66	(a - b) - c	a - (b + c)			16				16	
77	a - (b - c)	(a - b) + c		7				7		
88	(a ÷ b) ÷ c	a ÷ (b × c)	11				11			
99	a ÷ (b ÷ c)	(a ÷ b) × c				7				7

Table 1.5

Distribution of Inventory C Forms Among Pupils Tested

Seventh grade class	Form 1	Form 2	Form 3	Form 4	Total
	Number of pupils				
A	6	6	6	6	24
B	6	6	6	6	24
C	8	8	8	8	32
D	5	5	5	5	20
Total	25	25	25	25	100

Equivalence Inventory C

Earlier piloting of an instrument, Equivalence Inventory B (reproduced in Kasanda, 1977), led to its extension and refinement in accord with Tables 1.3 and 1.4, which was then administered to students from four classes as summarized in Table 1.5. Each of the 100 students completed one of the four randomly distributed Inventory forms (copies are included in the text of this report following Table 1.5).

Equivalence Inventory C

Form 1

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.☐ C

NAME ______________________________

SAMPLE PROBLEMS

S1. $4 \times 6520 = \boxed{n}$.

$40 \times 652 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

S2. $713 + 489 = \boxed{n}$.

$713 - 489 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

S3. $128 \times 896 = \boxed{n}$.

$896 : 128 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

S4. $(954 + 954) - 0 = \boxed{n}$.

$0 + (2 \times 954) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

STOP.

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PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.$\boxed{1}$ C

1. $472 + (85 \times 103) = \boxed{n}$.
$(472 + 85) \times 103 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

2. $(29 \times 206) - 157 = \boxed{n}$.
$29 \times (206 - 157) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

3. $563 + (129 + 487) = \boxed{n}$.
$(563 + 129) + 487 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

4. $(1176 - 294) \div 98 = \boxed{n}$.
$1176 - (294 \div 98) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

5. $1025 - (23 \times 41) = \boxed{n}$.
$(23 \times 41) - 1025 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

6. $(1128 : 94) + 282 = \boxed{n}$.
$1128 : (94 + 282) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

7. $806 - (759 - 481) = \boxed{n}$.
$(806 - 759) + 481 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

GO RIGHT ON TO THE NEXT PAGE.

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.1 C

8. $121 \times (57 : 19) = \boxed{n}$.

$(121 \times 57) : 19 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

9. $(915 - 274) + 638 = \boxed{n}$.

$915 - (274 + 638) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

10. $28 \times (35 + 469) = \boxed{n}$.

$(28 \times 35) + 469 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

11. $(1856 : 58) : 4 = \boxed{n}$.

$1856 : (58 \times 4) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

12. $(288 + 1152) \div 96 = \boxed{n}$.

$288 + (1152 \div 96) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

13. $(217 \times 41) \times 36 = \boxed{n}$.

$217 \times (41 \times 36) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

14. $1058 : (46 - 23) = \boxed{n}$.

$(1058 : 46) - 23 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

GO RIGHT ON TO THE NEXT PAGE.

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.$\boxed{1}$ C

15. 1015 - (26 × 34) = $\boxed{n}$.

(1015 - 26) × 34 = $\boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

16. (1176 : 56) + 479 = $\boxed{n}$.

479 + (1176 : 56) = $\boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

17. 862 - (537 - 294) = $\boxed{n}$.

(862 - 537) - 294 = $\boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

18. 3234 : (231 : 7) = $\boxed{n}$.

(3234 : 231) : 7 = $\boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

19. (632 + 714) - 598 = $\boxed{n}$.

632 + (714 - 598) = $\boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

20. (1512 : 21) × 18 = $\boxed{n}$.

1512 : (21 × 18) = $\boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

STOP. RAISE YOUR HAND SO THAT YOUR BOOKLET MAY BE COLLECTED.

Equivalence Inventory C

Form 2

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.☐ C

NAME ____

SAMPLE PROBLEMS

S1. $4 \times 6520 = \boxed{n}$.

$40 \times 652 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

S2. $713 + 489 = \boxed{n}$.

$713 - 489 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

S3. $128 \times 896 = \boxed{n}$.

$896 : 128 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

S4. $(954 + 954) - 0 = \boxed{n}$.

$0 + (2 \times 954) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

STOP.

DON'T TURN TO THE NEXT PAGE YET.

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PLEASE DO NOT WRITE IN THESE BOXES. ☐ ☐ ☐ ☐☐.2 C

1. $(28 \times 35) + 469 = \boxed{n}$.

$28 \times (35 + 469) = \boxed{s}$ Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

2. $(1015 - 26) \times 34 = \boxed{n}$.

$1015 - (26 \times 34) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

3. $217 \times (41 \times 36) = \boxed{n}$.

$(217 \times 41) \times 36 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

4. $915 - (274 + 638) = \boxed{n}$.

$(915 - 274) + 638 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

5. $(326 + 484) : 54 = \boxed{n}$.

$54 : (326 + 484) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

6. $(472 + 85) \times 103 = \boxed{n}$.

$472 + (85 \times 103) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

7. $1156 : (68 : 17) = \boxed{n}$.

$(1156 : 68) \times 17 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

GO RIGHT ON TO THE NEXT PAGE.

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.$\boxed{2}$ C

8. $632 + (714 - 598) = \boxed{n}$.

   $(632 + 714) - 598 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

9. $1176 - (294 \div 98) = \boxed{n}$.

   $(1176 - 294) \div 98 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

10. $(3234 \div 231) \div 7 = \boxed{n}$.

    $3234 \div (231 \div 7) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

11. $86 \times (703 - 264) = \boxed{n}$.

    $(703 - 264) \times 86 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

12. $(862 - 537) - 294 = \boxed{n}$.

    $862 - (537 - 294) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

13. $(121 \times 57) \div 19 = \boxed{n}$.

    $121 \times (57 \div 19) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

14. $1128 \div (94 + 282) = \boxed{n}$.

    $(1128 \div 94) + 282 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

GO RIGHT ON TO THE NEXT PAGE.

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.2 C

15. $1512 \div (21 \times 18) = \boxed{n}$.
$(1512 \div 21) \times 18 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

16. $(951 - 326) - 487 = \boxed{n}$.
$951 - (326 + 487) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

17. $288 + (1152 \div 96) = \boxed{n}$.
$(288 + 1152) \div 96 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

18. $(1058 \div 46) - 23 = \boxed{n}$.
$1058 \div (46 - 23) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

19. $(563 + 129) + 487 = \boxed{n}$.
$563 + (129 + 487) = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

20. $29 \times (206 - 157) = \boxed{n}$.
$(29 \times 206) - 157 = \boxed{s}$. Is $\boxed{s}$ the same number as $\boxed{n}$? SURE YES SURE NO NOT SURE

STOP. RAISE YOUR HAND SO THAT YOUR BOOKLET MAY BE COLLECTED.

Equivalence Inventory C

Form 3

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.☐ C

NAME ______________________________

SAMPLE PROBLEMS

S1. $4 \times 6520 = 40 \times 652$. SURE TRUE SURE FALSE NOT SURE

S2. $713 + 489 = 713 - 489$. SURE TRUE SURE FALSE NOT SURE

S3. $128 \times 896 = 896 \div 128$. SURE TRUE SURE FALSE NOT SURE

S4. $(954 + 954) - 0 = 0 + (2 \times 954)$. SURE TRUE SURE FALSE NOT SURE

STOP.

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PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.3 C

1. $472 + (85 \times 103) = (472 + 85) \times 103$. SURE TRUE SURE FALSE NOT SURE

2. $(29 \times 206) - 157 = 29 \times (206 - 157)$. SURE TRUE SURE FALSE NOT SURE

3. $563 + (129 + 487) = (563 + 129) + 487$. SURE TRUE SURE FALSE NOT SURE

4. $(1176 - 294) \div 98 = 1176 - (294 \div 98)$. SURE TRUE SURE FALSE NOT SURE

5. $1025 - (23 \times 41) = (23 \times 41) - 1025$. SURE TRUE SURE FALSE NOT SURE

6. $(1128 \div 94) + 282 = 1128 \div (94 + 282)$. SURE TRUE SURE FALSE NOT SURE

7. $806 - (759 - 481) = (806 - 759) + 481$. SURE TRUE SURE FALSE NOT SURE

8. $121 \times (57 \div 19) = (121 \times 57) \div 19$. SURE TRUE SURE FALSE NOT SURE

9. $(915 - 274) + 638 = 915 - (274 + 638)$. SURE TRUE SURE FALSE NOT SURE

10. $28 \times (35 + 469) = (28 \times 35) + 469$. SURE TRUE SURE FALSE NOT SURE

GO RIGHT ON TO THE NEXT PAGE.

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.3 C

11. $(1856 \div 58) \div 4 = 1856 \div (58 \times 4)$. SURE TRUE SURE FALSE NOT SURE

12. $(288 + 1152) \div 96 = 288 + (1152 \div 96)$. SURE TRUE SURE FALSE NOT SURE

13. $(217 \times 41) \times 36 = 217 \times (41 \times 36)$. SURE TRUE SURE FALSE NOT SURE

14. $1058 \div (46 - 23) = (1058 \div 46) - 23$. SURE TRUE SURE FALSE NOT SURE

15. $1015 - (26 \times 34) = (1015 - 26) \times 34$. SURE TRUE SURE FALSE NOT SURE

16. $(1176 \div 56) + 479 = 479 + (1176 \div 56)$. SURE TRUE SURE FALSE NOT SURE

17. $862 - (537 - 294) = (862 - 537) - 294$. SURE TRUE SURE FALSE NOT SURE

18. $3234 \div (231 \div 7) = (3234 \div 231) \div 7$. SURE TRUE SURE FALSE NOT SURE

19. $(632 + 714) - 598 = 632 + (714 - 598)$. SURE TRUE SURE FALSE NOT SURE

20. $(1512 \div 21) \times 18 = 1512 \div (21 \times 18)$. SURE TRUE SURE FALSE NOT SURE

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Equivalence Inventory C

Form 4

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.☐ C

NAME ______________________________

SAMPLE PROBLEMS

S1.	$4 \times 6520 = 40 \times 652$.	SURE TRUE	SURE FALSE	NOT SURE
S2.	$713 + 489 = 713 - 489$.	SURE TRUE	SURE FALSE	NOT SURE
S3.	$128 \times 896 = 896 \div 128$.	SURE TRUE	SURE FALSE	NOT SURE
S4.	$(954 + 954) - 0 = 0 + (2 \times 954)$.	SURE TRUE	SURE FALSE	NOT SURE

STOP.

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1. $(28 \times 35) + 469 = 28 \times (35 + 469)$. SURE TRUE | SURE FALSE | NOT SURE

2. $(1015 - 26) \times 34 = 1015 - (26 \times 34)$. SURE TRUE | SURE FALSE | NOT SURE

3. $217 \times (41 \times 36) = (217 \times 41) \times 36$. SURE TRUE | SURE FALSE | NOT SURE

4. $915 - (274 + 638) = (915 - 274) + 638$. SURE TRUE | SURE FALSE | NOT SURE

5. $(326 + 484) \div 54 = 54 \div (326 + 484)$. SURE TRUE | SURE FALSE | NOT SURE

6. $(472 + 85) \times 103 = 472 + (85 \times 103)$. SURE TRUE | SURE FALSE | NOT SURE

7. $1156 \div (68 \div 17) = (1156 \div 68) \times 17$. SURE TRUE | SURE FALSE | NOT SURE

8. $632 + (714 - 598) = (632 + 714) - 598$. SURE TRUE | SURE FALSE | NOT SURE

9. $1176 - (294 \div 98) = (1176 - 294) \div 98$. SURE TRUE | SURE FALSE | NOT SURE

10. $(3234 \div 231) \div 7 = 3234 \div (231 \div 7)$. SURE TRUE | SURE FALSE | NOT SURE

GO RIGHT ON TO THE NEXT PAGE.

PLEASE DO NOT WRITE IN THESE BOXES: ☐ ☐ ☐ ☐☐.4 C

11. 86 × (703 - 264) = (703 - 264) × 86. SURE TRUE SURE FALSE NOT SURE

12. (862 - 537) - 294 = 862 - (537 - 294). SURE TRUE SURE FALSE NOT SURE

13. (121 × 57) ÷ 19 = 121 × (57 ÷ 19). SURE TRUE SURE FALSE NOT SURE

14. 1128 ÷ (94 + 282) = (1128 ÷ 94) + 282. SURE TRUE SURE FALSE NOT SURE

15. 1512 ÷ (21 × 18) = (1512 ÷ 21) × 18. SURE TRUE SURE FALSE NOT SURE

16. (951 - 326) - 487 = 951 - (326 + 487). SURE TRUE SURE FALSE NOT SURE

17. 288 + (1152 ÷ 96) = (288 + 1152) ÷ 96. SURE TRUE SURE FALSE NOT SURE

18. (1058 ÷ 46) - 23 = 1058 ÷ (46 - 23). SURE TRUE SURE FALSE NOT SURE

19. (563 + 129) + 487 = 563 + (129 + 487). SURE TRUE SURE FALSE NOT SURE

20. 29 × (206 - 157) = (29 × 206) - 157. SURE TRUE SURE FALSE NOT SURE

STOP. RAISE YOUR HAND SO THAT YOUR BOOKLET MAY BE COLLECTED.

The Findings

Table 1.6 indicates that there was no cause to reject the null hypothesis[1]

$$H_0: \mu_A = \mu_B = \mu_C = \mu_D.$$

Consequently, data were pooled across classes for all subsequent analyses.

Inventory Forms

Table 1.7 summarizes the tests for 6 null hypotheses, $\mu_X = \mu_Y$, none of which was rejected.[1] Forms 1 and 2 were thus considered to be comparable statistically as well as logically with respect to mean performance on the 16-item subtest, the 4-item subtest, and the 20-item *Inventory*. A similar statement can be made for Forms 3 and 4.

In subsequent analyses, data were pooled across Forms 1 and 2 for the two-sentence level of the phrase format factor, and across Forms 3 and 4 for the one-sentence level of the phrase format factor.

Reliability estimates for each Form of the 20-item Inventory and the 16-item subtest are summarized in Table 1.8. However, no such estimates were determined for the 4-item subtest.

Principal Analyses

An overview of student performance on *Equivalence Inventory C*, without regard for Table 1.1 phrase types or Table 1.2 factors and levels, is presented in Tables 1.9 and 1.10. It is evident from Table 1.9 that for each of the 3-item sets--the 16-item subtest, the 4-item subtest, and the 20-item *Inventory*--the mean number of correct responses was less than 50% of the items in that particular item set.

---

[1] At $\alpha = .10$

Table 1.6

Summary of Performance by Seventh Grade Classes on Inventory C

Class	Mean correct responses	Variance table				
		Source of variation	SS	df	MS	F
A	8.29					
B	9.00	Between classes	31.092	3	10.364	.862
C	9.25	Within classes	1153.908	96	12.020	
D	9.95	Total	1185.000	99		

Notes. $\bar{G} = 9.10$

$F_{.90}(3,96) = 2.15$

Table 1.7

Summary of Performance (Correct Responses) on Logically-comparable *Inventory C* Forms

Item set	Form 1	Form 2	*F*	Form 3	Form 4	*F*
	Mean			Mean		
16-item subtest	8.64	8.52	.02	7.12	6.24	.94
4-item subtest	1.20	1.56	2.42	1.60	1.52	.11
20-item *Inventory*	9.84	10.08	.07	8.72	7.76	.94

Note. $F_{.90}(1,48) = 2.82$

Table 1.8

ANOVAs for Reliability

Source of variation	SS	df	MS	r
20-item Form 1				
Between pupils	11.568	24	.482	.56
Residual	96.752	456	.212	
16-item Form 1				
Between pupils	12.360	24	.515	.57
Residual	79.000	360	.219	
20-item Form 2				
Between pupils	13.892	24	.579	.65
Residual	91.228	456	.200	
16-item Form 2				
Between pupils	16.515	24	.688	.70
Residual	74.525	360	.207	
20-item Form 3				
Between pupils	20.652	24	.861	.78
Residual	86.868	456	.191	
16-item Form 3				
Between pupils	19.415	24	.809	.77
Residual	68.025	360	.189	
20-item Form 4				
Between pupils	8.828	24	.368	.48
Residual	88.052	456	.193	
16-item Form 4				
Between pupils	11.410	24	.475	.58
Residual	71.390	360	.198	

Note. In each instance $r = \frac{MS_{b.pupils} - MS_{residual}}{MS_{b.pupils}}$

Table 1.9

Overview of Pupil Performance on Equivalence Inventory C

Item set	SURE YES / SURE NO or SURE TRUE / SURE FALSE responses		NOT SURE responses	All other responses
	Mean correct	Mean not-correct	Mean	
16-item subtest: phrases 1 thru 16	7.63	6.91	1.41	.05
4-item subtest: other phrase combinations	1.47	2.30	.22	.01
20-item Inventory	9.10	9.21	1.63	.06

Table 1.10

Frequency Distributions and Variance Estimates

Number of correct responses	16-item subtest	4-item subtest	20-item Inventory
	Number of pupils		
20	X	X	
19	X	X	1
18	X	X	
17	X	X	
16		X	2
15	2	X	2
14	1	X	4
13	2	X	10
12	11	X	7
11	7	X	9
10	8	X	12
9	10	X	12
8	9	X	5
7	8	X	6
6	10	X	11
5	8	X	12
4	18	2	5
3	3	8	1
2	2	33	
1	1	49	1
0		8	
s^2	10.66	.70	11.97

Note. X signifies that no score of a particular magnitude exists for a certain subtest.

Table 1.10 gives evidence of a nontrivial degree of variability of performance in terms of correct responses on each of the 3-item sets.

It should also be noted in Table 1.9 that the mean number of "NOT SURE" and of "All other" responses was relatively small. Such responses were not the principal reason for the relatively low number of mean correct responses.

1. Primary null hypotheses. Three null hypotheses were tested for the 20-item *Inventory C*, for the 16-item subtest, and for the 4-item subtest:

a. There is no significant difference ($\alpha = .10$) between the mean number of correct responses on the two levels of item format: one-sentence vs. two-sentence.

b. There is no significant difference ($\alpha = .10$) between the mean number of correct responses on the two levels of parentheses positions: first left, then right vs. first right, then left.

c. There is no significant interaction ($\alpha = .10$) in terms of correct-response performance between the two preceding factors.

In each instance the three hypotheses were tested by a 2 x 2 ANOVA involving a fixed effects model with repeated measures on the second factor.

1.1. The 20-item *Inventory C*. Cell and marginal means for correct responses are summarized in Table 1.11, with the subsequent ANOVA summarized in Table 1.12. The data from Table 1.12 give cause to reject each of the three null hypotheses for the 20-item *Inventory C*.

Table 1.11

Cell and Marginal Means for Correct Responses

for 20-item Inventory C ANOVA

A. Item format	B. Parentheses positions		Across B
	b_1 Left/Right(LR)	b_2 Right/Left(RL)	
a_1 One-sentence	4.10	4.14	4.12
a_2 Two-sentence	4.58	5.38	4.98
Across A	4.34	4.76	4.55

Note. Each cell mean pertains to a set of 10 items.

Table 1.12

ANOVA for Correct Responses on 20-item Inventory C

Source of variation	SS	df	MS	F
Between pupils	592.500	99		
A (item format)	36.980	1	36.980	6.524##
Pupils within groups	555.520	98	5.669	
Within pupils	213.000	100		
B (parentheses positions)	8.820	1	8.820	4.389*
AB	7.220	1	7.220	3.592#
B × pupils within groups	196.960	98	2.010	
Total	805.500	199		

$F_{.975}(1,98) < F_{obs} < F_{.99}(1,98)$

* $F_{.95}(1,98) < F_{obs} < F_{.975}(1,98)$

# $F_{.90}(1,98) < F_{obs} < F_{.95}(1,98)$

1.2. The 16-item subtest. Cell and marginal means for correct responses are summarized in Table 1.13, with the subsequent ANOVA summarized in Table 1.14.

The data from Table 1.14 give cause to reject the two null hypotheses pertaining to main effects (item format and parentheses positions), but there is not cause to reject the null hypotheses pertaining to an interaction effect for the 16-item subtest.

1.3. The 4-item subtest. Data from Tables 1.15 and 1.16 give no cause to reject the null hypotheses pertaining to either main or interaction effects.

A sub-analysis: Associativity/Nonassociativity

Although evidence pertaining to various groups of student's comprehension of associativity of addition and multiplication has been gathered by a variety of persons (Kasanda, 1977), little evidence has been reported regarding students' sensitivity to the nonassociativity of subtraction and division. Some of this information was embedded within the present investigation, however, and is summarized in Tables 1.17, 1.18, 1.19, 1.20, 1.21, and 1.22, and in Figure 1.1. For the sample involved in this investigation:

1. awareness of associativity of addition and multiplication was significantly higher than awareness of the nonassociativity of subtraction and division, although such awareness was associated significantly with the item-format factor;

2. no significant difference was observed between awareness of associativity of addition and associativity of multiplication; however, a significant difference was observed between awareness of nonassociativity of subtraction and nonassociativity of division.

Table 1.13

Cell and Marginal Means of Correct Responses

for 16-item Subtest ANOVA

A. Item format	B. Parentheses positions		Across B
	b_1 Left/Right(LR)	b_2 Right/Left(RL)	
a_1 One-sentence	3.30	3.38	3.34
a_2 Two-sentence	4.00	4.58	4.29
Across A	3.65	3.98	3.82

Note. Each cell mean pertains to a set of 8 items.

Table 1.14

ANOVA for Correct Responses on 16-item Subtest

Source of variation	SS	df	MS	F
Between pupils	527.655	99		
A (item format)	45.125	1	45.125	9.165**
Pupils within groups	482.530	98	4.924	
Within pupils	148.500	100		
B (parentheses positions)	5.445	1	5.445	3.813#
AB	3.125	1	3.125	2.189
B × pupils within groups	139.930	98	1.428	
Total	676.155	199		

** $F_{obs} > F_{.99}(1,98)$

$F_{.90}(1,98) < F_{obs} < F_{.95}(1,98)$

Table 1.15

Cell and Marginal Means of Correct Responses

for 4-item Subtest ANOVA

A. Item format	B. Parentheses positions		Across B
	b_1 Left/Right(LR)	b_2 Right/Left(RL)	
a_1 One-sentence	.80	.76	.78
a_2 Two-sentence	.58	.80	.69
Across A	.69	.78	.74

Note. Each cell mean pertains to a set of 2 items.

Table 1.16

ANOVA for Correct Responses on 4-item Subtest

Source of variation	SS	df	MS	F
Between pupils	34.455	99		
A (item format)	.405	1	.405	1.166
Pupils within groups	34.050	98	.347	
Within pupils	42.500	100		
B (parentheses positions)	.405	1	.405	.962
AB	.845	1	.845	2.008
B × pupils within groups	41.250	98	.421	
Total	76.955	199		

Note. In each instance $F_{obs} < F_{.90}(1,98)$

Table 1.17

Cell and Marginal Means (Correct Responses) on Associative Property Items

A. Item format	B. Operations		Across B
	b_1 Addition and multiplication	b_2 Subtraction and division	
a_1 One-sentence	1.56	.60	1.08
a_2 Two-sentence	1.58	.96	1.27
Across A	1.57	.78	1.18

Note. Each cell mean pertains to a set of 2 items.

Table 1.18

ANOVA for Correct Responses on Associative Property Items

Source of variation	SS	df	MS	F
Between pupils	30.375	99		
A (item format)	1.805	1	1.805	6.191##
Pupils within groups	28.570	98	.292	
Within pupils	96.500	100		
B (operations)	31.205	1	31.205	47.895***
AB	1.445	1	1.445	2.218
B × pupils within groups	63.850	98	.652	
Total	126.875	199		

$F_{.975}(1,98) < F_{obs} < F_{.99}(1,98)$

*** $F_{obs} > F_{.999}(1,98)$

Otherwise, $F_{obs} < F_{.90}(1,98)$

Table 1.19

Cell and Marginal Means (Correct Responses)

for Phrases 1 and 11

A. Item format	B. Operation		Across B
	b_1 Addition	b_2 Multiplication	
a_1 One-sentence	.74	.82	.78
a_2 Two-sentence	.88	.70	.79
Across A	.81	.76	.785

Table 1.20

ANOVA for Correct Responses on Phrases 1 and 11

Source of variation	*SS*	*df*	*MS*	*F*
Between pupils	18.255	99		
A (item format)	.005	1	.005	.027
Pupils within groups	18.250	98	.186	
Within pupils	15.500	100		
B (operation)	.125	1	.125	.843
AB	.845	1	.845	5.699##
B × pupils within groups	14.530	98	.148	
Total	33.755	199		

$F_{.975}(1,98) < F_{obs} < F_{.99}(1,98)$

Otherwise, $F_{obs} < F_{.90}(1,98)$

Table 1.21

Cell and Marginal Means (Correct Responses)

for Phrases 6 and 16

A. Item format	B. Operation		Across B
	b_1 Subtraction	b_2 Division	
a_1 One-sentence	.26	.34	.30
a_2 Two-sentence	.42	.54	.48
Across A	.34	.44	.39

Table 1.22

ANOVA for Correct Responses on Phrases 6 and 16

Source of variation	SS	df	MS	F
Between pupils	29.580	99		
A (item format)	1.620	1	1.620	5.678##
Pupils within groups	27.960	98	.285	
Within pupils	18.000	100		
B (operation)	.500	1	.500	2.803#
AB	.020	1	.020	.112
B × pupils within groups	17.480	98	.178	
Total	47.580	199		

$F_{.975}(1,98) < F_{obs} < F_{.99}(1,98)$

$F_{.90}(1,98) < F_{obs} < F_{.95}(1,98)$

Otherwise, $F_{obs} < F_{.90}(1,98)$

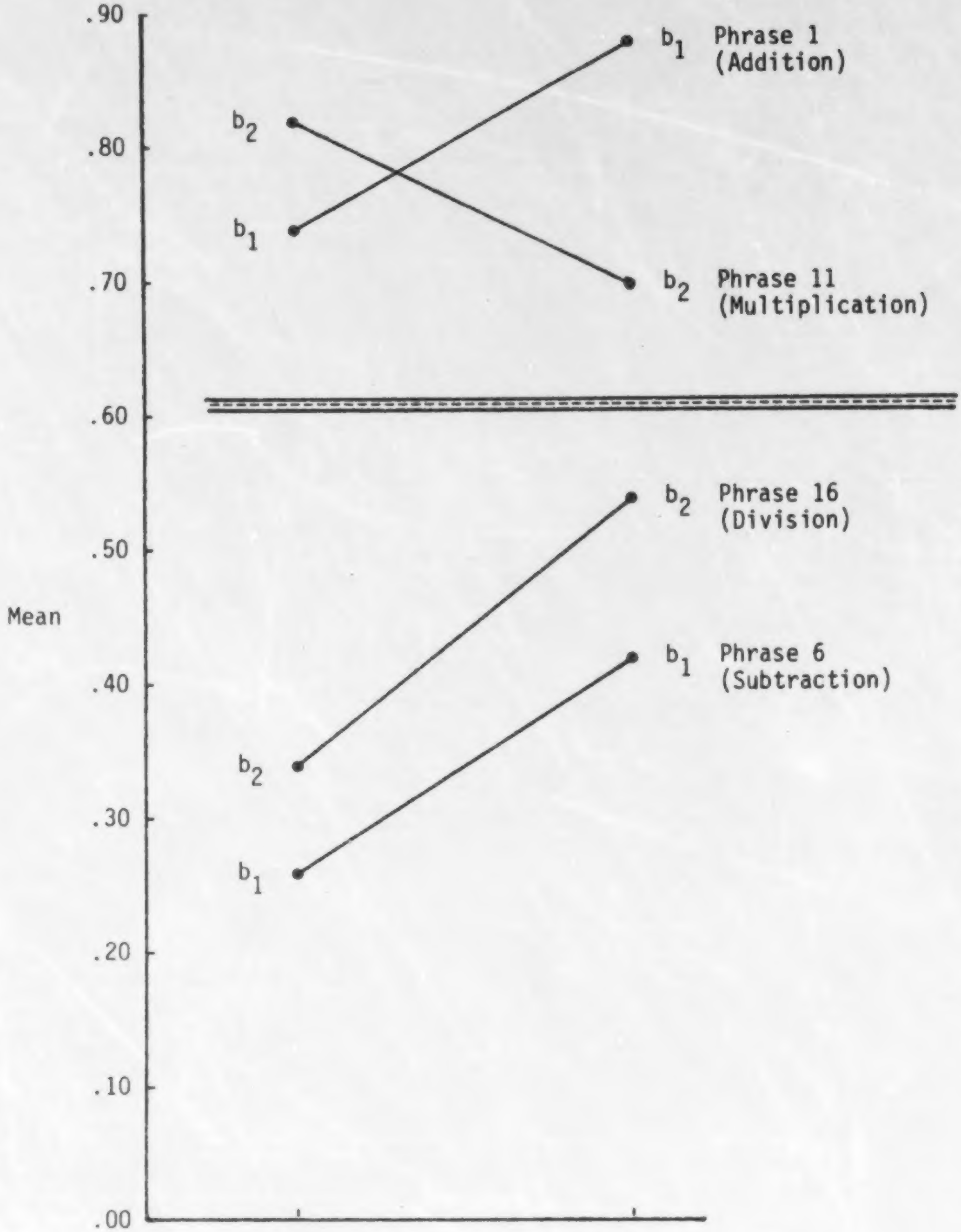

Figure 1.8. Graph of cell means (correct responses) for phrases 1 and 11 and for phrases 6 and 16 (associativity items).

Tables 1.23 and 1.24 summarize findings by phrase type rather than by factors and levels.

Subsequent Instructional Observations

A subset of the 100-student sample consisted of students who were receiving systematic weekly instruction pertaining to a variety of uses of electronic calculators, in particular the Litronix models 2260 (parentheses but no "memory") and 2250 ("memory" but no parentheses) as identified in the Introduction.

It was both feasible and desirable to use such calculators in an attempt to clarify some of the students' misconceptions that could be inferred from Inventory C findings. The effect of instruction designed for this purpose was not assessed in any formal way. However, one important and concluding observation should be noted.

When working with exemplars of the form "a * (b # c)," students found it much simpler to use model 2260 calculators (with parentheses) than to use model 2250 calculators (with "memory"). Calculations with the 2260's were very direct and straightforward, but many of the same or similar calculations with the 2250's were more roundabout to the students in the sense that data frequently had to be "manipulated" in relation to the calculator's "memory" (see Table 1.25). It seemed to be more difficult for students using the 2250's to generate (and recall) generic calculator algorithms than when using the 2260's.

Equivalence Inventory C is believed to be an instrument that can be used to advantage as a means of identifying certain instructional needs. Electronic calculators of a suitable type can be helpful to students when confronted with exemplars pertaining to the equivalence or nonequivalence of number phrases and number sentences.

A. Item format
1. 1-sentence
2. 2-sentence

B. Parentheses
1. LR
2. RL

Table 1.23

Summary of ANOVAs for Phrase Types

Phrase type	Cell mean				Marginal mean					Observed $\underline{F}$		
	ab_{11}	ab_{12}	ab_{21}	ab_{22}	A_1	A_2	B_1	B_2	G	A	B	AB
1. a + b + c	.68	.80	.80	.96	.74	.88	.74	.88	.81	3.27#	3.27#	.07
2. a + b - c	.64	.60	.56	.80	.62	.68	.60	.70	.65	.39	1.09	2.15
3. a + b × c	.16	.36	.32	.44	.26	.38	.24	.40	.32	1.67	2.97#	.19
4. a + b ÷ c	.32	.48	.56	.36	.40	.46	.44	.42	.43	.37	.04	3.29#
5. a - b + c	.40	.24	.40	.52	.32	.46	.40	.38	.39	2.06	.04	2.06
6. a - b - c	.36	.16	.36	.48	.26	.42	.36	.32	.34	2.91#	.18	2.91#
7. a - b × c	.24	.40	.48	.48	.32	.48	.36	.44	.40	2.67	.67	.67
8. a - b ÷ c	.48	.32	.44	.60	.40	.52	.46	.46	.46	1.45	.00	2.58
9. a × b + c	.24	.32	.44	.36	.28	.40	.34	.34	.34	1.58	.00	.70
10. a × b - c	.40	.36	.44	.60	.38	.52	.42	.48	.45	1.97	.36	1.00
11. a × b × c	.80	.84	.64	.76	.82	.70	.72	.80	.76	1.95	.87	.22
12. a × b ÷ c	.40	.48	.52	.44	.44	.48	.46	.46	.46	.16	.00	.62
13. a ÷ b + c	.48	.32	.64	.52	.40	.58	.56	.42	.49	3.28#	1.99	.04
14. a ÷ b - c	.24	.44	.40	.68	.34	.54	.32	.56	.44	4.33*	6.24##	.17
15. a ÷ b × c	.36	.36	.52	.56	.36	.54	.44	.46	.45	3.25#	.04	.04
16. a ÷ b ÷ c	.40	.28	.48	.60	.34	.54	.44	.44	.44	4.12*	.00	1.49
22/33.	.72	.88	.60	.92	.80	.76	.66	.90	.78	.25	8.91**	.99
44/55.	.28	.40	.28	.36	.34	.32	.28	.38	.33	.04	1.10	.04
66/77.	.32	.20	.12	.08	.26	.10	.22	.14	.18	4.41*	1.10	.28
88/99.	.28	.04	.16	.24	.16	.20	.22	.14	.18	.28	1.10	4.41*

$\underline{F}_{.90}(1,96) < \underline{F}_{obs} < \underline{F}_{.95}(1,96)$

* $\underline{F}_{.95}(1,96) < \underline{F}_{obs} < \underline{F}_{.975}(1,96)$

$\underline{F}_{.975}(1,96) < \underline{F}_{obs} < \underline{F}_{.99}(1,96)$

** $\underline{F}_{obs} > \underline{F}_{.99}(1,96)$

Otherwise,
$\underline{F}_{obs} < \underline{F}_{.90}(1,96)$

Table 1.24

Rank Order of Phrase Types Based Upon Mean Correct Responses

Rank	Phrase type	Mean
1	1. a + b + c	.81
2	22. (a ÷ b) + c, c + (a ÷ b) 33. a × (b - c), (b - c) × a	.78
3	11. a × b × c	.76
4	2. a + b - c	.65
5	13. a ÷ b + c	.49
6.5	8. a - b ÷ c	.46
6.5	12. a × b ÷ c	.46
8.5	10. a × b - c	.45
8.5	15. a ÷ b × c	.45
10.5	14. a ÷ b - c	.44
10.5	16. a ÷ b ÷ c	.44
12	4. a + b ÷ c	.43
13	7. a - b × c	.40
14	5. a - b + c	.39
15.5	6. a - b - c	.34
15.5	9. a × b + c	.34
17	44. (a + b) ÷ c, c ÷ (a + b) 55. a - (b × c), (b × c) - a	.33
18	3. a + b × c	.32
19.5	66. (a - b) - c, a - (b + c) 77. a - (b - c), (a - b) + c	.18
19.5	88. (a ÷ b) ÷ c, a ÷ (b × c) 99. a ÷ (b ÷ c), (a ÷ b) × c	.18

Table 1.25

Algorithms for Litronix Calculators

(a * b) # c 2260 or 2250

INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
a	*				
b	#				a * b
c	=				(a * b) # c

a * (b # c) 2260

INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
a	*	(			
b	#				
c	)				b # c
	=				a * (b # c)

a * (b # c) 2250

INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
a	M	=			
b	#				
c	=				b # c
	M	*	RM		a * (b # c)

a * (b # c) 2250 (Alternative)

INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
b	#				
c	=	M	=		b # c
a	*	RM	=		a * (b # c)

Reference

Kasanda, C. D. *How well do seventh grade students perform on tasks involving the associative principle?* Unpublished masters thesis, The University of Wisconsin-Madison, 1977.

II

STUDENT PERFORMANCE ON OPEN SENTENCES INVOLVING THE DOING/UNDOING PROPERTY

by

Glendon W. Blume and Charles E. Mitchell

Introduction

When a pattern, property, or theorem in mathematics is taught, students may often attempt to apply this result to examples in which it does not hold. An example of this might be forming the generalization that all operations are commutative after learning that addition and multiplication of real numbers are commutative. As a further example, teachers often encounter students who, after correctly applying cancellation of factors in a rational expression $\frac{a \times b}{b}$, try to apply the same procedure to $\frac{a + b}{b}$.

Purpose

The purpose of this study was to assess seventh grade students' recognition and application of the "doing-undoing" pattern for the solution of a variety of open sentences such as 887 - 299 + 299 = N and (674 + 347) x 347 = N. The "doing-undoing" pattern in its purest form refers to examples such as the two examples above in which the second operation performed immediately cancels the effect of the first operation performed. Additional examples of the "doing-undoing" pattern, at times referred to as the pattern, are specified in the Research Design and Prodedure section. After identifying a population which demonstrated mastery of the use of the pattern, an attempt was made to identify members of this population who applied the pattern to open sentences in which it did not hold.

Included in this study is a further investigation of whether, in the absence of instruction which directly addressed this aspect, a commuted form of the pattern would be recognized by the students as a variation of the pattern.

Rationale

The topic of investigation of the use of "doing-undoing" in the solution of certain types of open sentences is a small part of the general area of interest in the importance of open sentences within the school mathematics curriculum. Since "doing-undoing" is an illustration of both the inverse relationship between operations and a relatively simple solution process for certain open sentences, it is a worthwhile area for exploratory research.

No previous research was found which addressed itself directly to the use of "doing-undoing" in open sentences. However, several studies, among them Grouws (1971), McMaster (1975), and Weaver (1971), indicate that performance on open sentence tasks is influenced by the size of the constants, the position of the placeholder, the type of operation in the open sentence, and the use of symmetric forms of the same sentence. Since little relevant research and no instrumentation was available, we have attempted what we consider a small step towards organized research on open sentences and the use of the "doing-undoing" pattern.

What a student has learned may not be identical to the learnings which were intended as an outcome of instruction. It may not be the case that the knowledge a student has acquired is simply a subset of the set of intended learnings, but that the student may have learned something (either desirable or undesirable) which was not an intended outcome of the instruction. A great deal of attention has been given to the question of determining the amount of intended learning that has occurred as a result of instruction, but little has been done to determine what unintended learning has occurred.

In particular, if students are taught to recognize the "doing-undoing" pattern, the question can be raised whether they will misapply that pattern to open sentences in which it does not hold (non-examples of the pattern). Such misapplication would represent a demonstration of an undesirable, unintended learning.

This investigation can also provide some information concerning teaching practices. Cooney, Davis, and Henderson (1975) and Shumway (1974) suggest that use of non-examples is important in teaching a generalization. The results of this study can shed some light on the necessity for a teacher to be concerned with providing non-examples of the "doing-undoing" pattern during the course of instruction on the pattern.

Research Design and Procedure

The population for the study consisted of 71 seventh grade students in three mathematics classes (sizes 28, 23, and 20 students) at Jefferson Middle School, Madison, Wisconsin. The population for the study was being used in a larger study and was selected because of its availability. Because of incomplete data gathered from one student, this student's scores were not reported in the results.

Design

The design of the study included an observation tailored to assess intended and unintended outcomes of prior instruction. We chose this pre-experimental (X O) design because a pre-test was impossible (instruction had already begun during the design of the study) and an approriate control group was not available.

Treatment

Each class received 11 half-hour instructional sessions in which hand-held calculators were used by each student in the class. Of these weekly sessions, four were designed to teach the "doing-undoing" pattern and the remaining sessions were concerned with unrelated topics using the hand-held calculator. The fourth "doing-undoing" session reviewed the work of the earlier three and in the following weekly session, the tests described below were administered.

The instruction in one class was provided by Professor J. F. Weaver in the presence of the two regular classroom teachers. The teachers then taught the same lessons using the same materials in the two remaining classes.

Test Description

Test 1. The first test, a copy of which is included in Appendix 2A, consisted of 20 randomly ordered items which were instances of the "doing-undoing" pattern in its purest form. Ten of these items were examples of addition-subtraction problems and 10 were examples of multiplication-division problems. Five items of each type had parentheses and 5 did not.

All items in Test 1 were similar to those used in class instruction. Numbers used in each problem were assumed to be large enough to deter students from making mental computations. Paper-and-pencil computations were not allowed.

An answer format was used with which the students may have been unfamiliar. For this reason, instructions were given prior to testing on the correst uasage of the answer format. Several unrelated sample problems were worked. The sample problems had no relationship to the pattern. At this time, all other instructions concerning the tests were stated.

Test 2. Test 2 is also included in Appendix 2A. Of the 20 items, four were similar to problems in Test 1, with two problems involving addition-subtraction, and two involving multiplication-division. Of these two sets, one problem in each had parentheses and one did not. These four problems were 4, 9, 15, and 18. These were included to provide some instances of the pattern.

Problems 3, 6, 12, and 17, hereafter referred to as the commutativity problems, were examples of the pattern, but in a commuted, rather than the pure form. Mastery of these problems was demonstrated by correctly answering at least three of the problems.

Problems 1, 10, 13, and 20, hereafter referred to as the parentheses problems, were non-examples of the pattern. Placement of the parentheses prevented the two operations from having inverse effects in the problem. Two of the problems used addition-subtraction and two used multiplication-division. Mastery of these problems was demonstrated by answering three or four questions correctly.

There were eight problems that used pairs of operations that would not have inverse effects on each other. These problems, hereafter referred to as the mixed signs problems, were numbers 2, 5, 7, 8, 11, 14, 16, and 19. Problems 8 and 16 paired subtraction-multiplication, 2 and 14 paired addition-division, 5 and 11 paired division-subtraction, and 7 and 19 paired multiplication-addition. For each pair of operations, the order of the operations was permuted to cancel any effects of order. Mastery was demonstrated by correctly answering seven or more problems.

There were 16 problems in Test 1 which could be answered by circling a specific number in the answer format and four problems that required a student to circle "some other number" in order to correctly answer the question. To achieve a mastery score on the first test, a student had to answer correctly 14 out of 16 on the first group and 3 out of 4 on the second group.

On the second test, an attempt was made to identify responses which represented misapplication of the pattern. A misapplication of the pattern was defined to be circling the number that would answer the problem if the two operations had inverse effects on each other. If the students misapplied the pattern at least three our of four times on the parentheses problems, or at least five out of eight times on the mixed signs problems, he or she was identified as an individual who was misapplying the pattern.

Test 2 was administered immediately after Test 1. Students had 5 minutes to complete each test. The test was administered to one class in the presence of regular teachers and the regular teachers administered the tests to the other two classes.

Data Analysis

An overall mean and standard deviation were computed for Test 1. In addition, the percentage of students meeting the mastery criterion on Test 1 was computed. For those students who had mastered Test 1, percentages of students in each of the five performance categories on Test 2 were determined: mastery of commutativity problems, mastery of mixed signs problems, mastery of parentheses problems, misapplying the pattern on mixed signs problems, misapplying the pattern on parentheses problems.

Since it was not possible to pre-test the class and since it was not feasible that a control group be used, most standard statistical analyses were not appropriate. Scatter plots were constructed for individual classes as well as for the entire population to investigate a possible relationship between performances on Test 1 and the various subtests of Test 2. A phi-coefficient was computed to measure association between mastery on Test 1 and mastery on Test 2.

Results

The mean score on Test 1 was 14.6 with a standard deviation of 5.6. As indicated in Table 2.1, 46% of the population demonstrated mastery of the "doing-undoing" pattern.

Of the 32 subjects who mastered the first test, 3 misapplied the pattern on the mixed signs problems, 4 misapplied the pattern on the parentheses problems, and 1 misapplied the pattern on both.

There were 25 subjects who mastered the commutativity problems, 22 subjects who mastered the mixed signs problems, and 21 subjects who mastered the parentheses problems. There were 19 subjects who mastered both the parentheses and mixed signs problems.

The information concerning mastery of Test 2 items for those subjects who did not master Test 1 is reported in Table 2.1 for purposes of comparison. Since they had not mastered the pattern, however, it was not meaningful to see if they were misapplying it.

Table 2.2 summarizes performances on Test 1 vs. Test 2. One unexpected result, which will be discussed later, was the nine subjects who mastered the mixed signs problems on Test 2 but did not master Test 1.

Table 2.1

Percentages of Students Demonstrating Various Types of Mastery and Misapplication of the Pattern

Test 1	Mastery of Pattern 46% (*n* = 32)			Non-mastery of Pattern 54% (*n* = 38)	
Test 2	Mastery on Mixed Signs 69% (*n* = 22)	Neither Mastery nor Misapplication 22% (*n* = 7)	Misapplication or Mixed Signs 9% (*n* = 3)	Mastery on Mixed Signs 24% (*n* = 9)	Non-mastery on Mixed Signs 76% (*n* = 29)
	Mastery on Parentheses 66% (*n* = 21)	Neither Mastery nor Misapplication 22% (*n* = 7)	Misapplication on Parentheses 12% (*n* = 4)	Mastery on Parentheses 42% (*n* = 16)	Non-mastery on Parentheses 58% (*n* = 22)
	Mastery on Commutativity 78% (*n* = 25)	Non-mastery on Commutativity 22% (*n* = 7)		Mastery on Commutativity 42% (*n* = 16)	Non-mastery on Commutativity 58% (*n* = 22)

Table 2.2

Contingency Tables Relating Mastery of Test 1 to Mastery of the Subtests of Test 2 (N = 70)

M = Mastery
N = Non-mastery
MIXED =Mixed Signs Problems
PAREN =Parentheses Problem
TOTAL =Mixed + Paren

Test 2, Mixed Signs

Test 1	M	N
M	22	10
N	9	29

Test 2, Parentheses

Test 1	M	N
M	21	11
N	16	22

Test 2, Total

Test 1	M	N
M	19	13
N	5	33

A phi-coefficient was computed as a measure of association between total Test 2 performance and Test 1 mastery as indicated in the third subtable of Table 2.2. A value of .49 was obtained, explaining 24% of the variance. The test statistic $\phi^2 N = 16.08$ was compared to χ_1^2 (.95) and was significant ($p < .01$).

Figures 2.1, 2.2, 2.3, and 2.4 are scatter plots. Individual classes are reported for both the parentheses and mixed signs problems. Included for each of the two problem sets are scatter plots for the total population.

Test 1 had a reliability coefficient of .90, using a technique similar to the split-half method. However, this coefficient was computed only for one of the three classes in the sample.

Interpretations

Since this study was exploratory in nature, the results from this pilot work must be interpreted in light of their implications for further investigation.

Conclusions

While the results are not conclusive, there is some evidence that a small percentage of the population will misapply a pattern such as the one used in this study. There was further evidence to indicate that while a specific student may have correctly answered most of the non-examples, when he or she missed a problem, it was due to misapplying the pattern.

In general, it does not appear that the instruction interfered with a student's prior knowledge of commutativity. For the most part, students correctly applied the property of commutativity when possible.

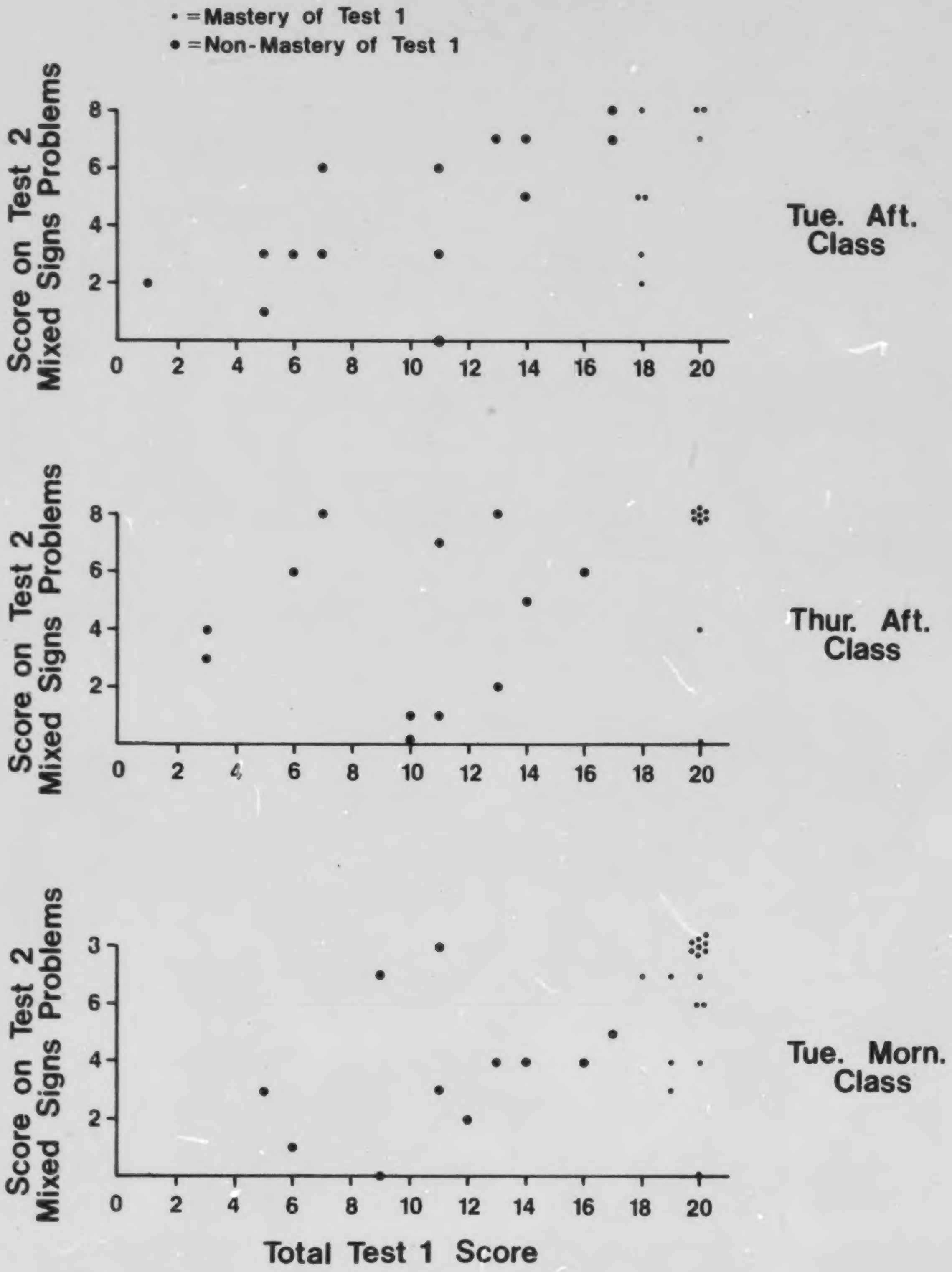

Figure 2.1. Scatter plots for Test 1 score vs. score on mixed signs problems.

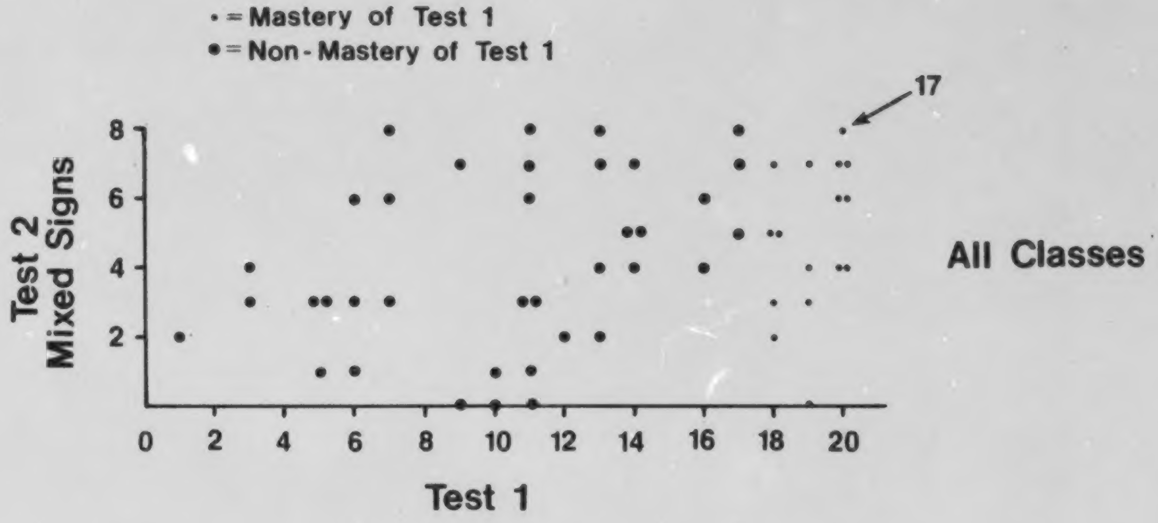

Figure 2.2. Cumulative results from Table 3.

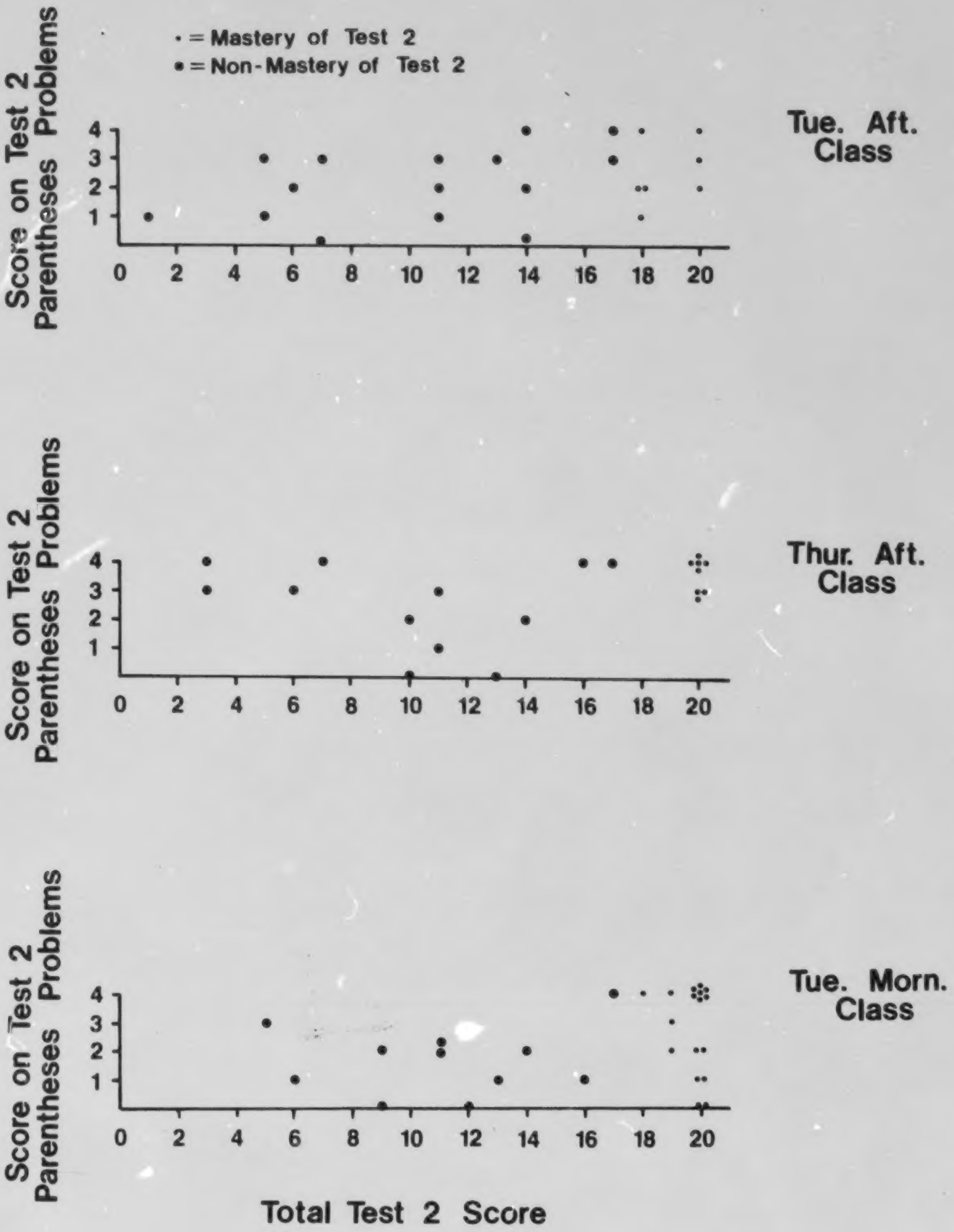

Figure 2.3. Scatter plots for Test 1 score vs. score on parentheses problems.

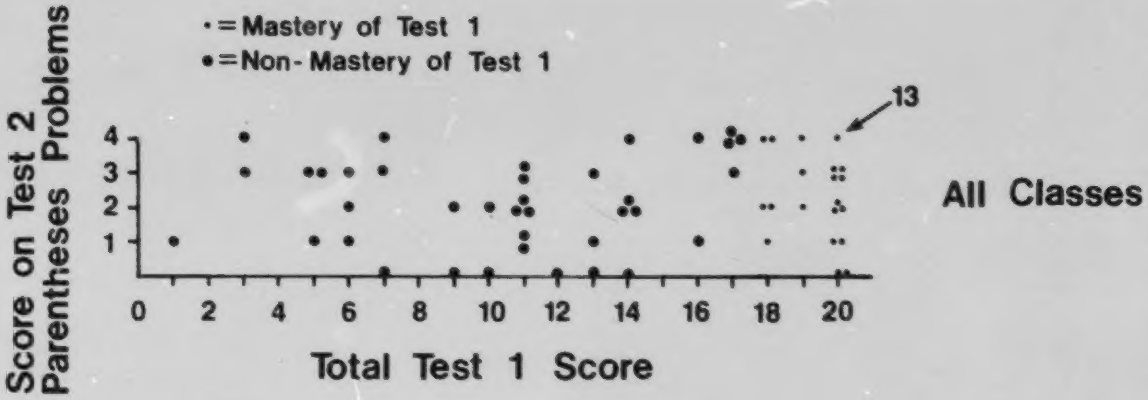

Figure 2.4. Cumulative results from Table 4.

Because of the absence of a pretest, it is difficult to explain why 69% of the students who mastered the first test also mastered the mixed signs problems. Of course, we would like to claim that this high percentage was due, at least in part, to the instruction. This might provide evidence pertaining to the usefulness or necessity of non-examples.

Since instruction was provided concerning the parentheses problems, we had expected a higher percentage to master these problems than the mixed signs problems. It may be true that students are more easily led to believe that the pattern holds in parentheses problems, or it may be more obvious that the pattern does not hold in mixed signs problems. This may also explain why almost the same percentage of students misapplied both non-examples.

The most surprising result was the group of nine students who did not master Test 1, yet mastered mixed signs problems in Test 2. After applying a t-test to the mean score (12.8) of these students and the mean score (9.7) of all other students who did not master Test 1, a significant difference in mean scores was found. Because the nine students had a significantly higher mean score at the .01 level, it was hypothesized that they may have had more knowledge of the mixed signs problems. Proficiency at a level closer to mastery on the first test may have had a major effect on performance on the mixed signs problems.

Limitations of the Study

The design of this study was admittedly limited; we suggest several improvements for further studies. While a control group could provide some information, it does not seem reasonable to use a control group since we would be studying unintended outcomes of instruction and such a control group would not be receiving instruction. However, it seems most desirable to pretest the sample to determine what knowledge students have prior to instruction.

Although indications have been given that the test was reliable, various improvements in the instrument are probably necessary. For example, on Test 1, in 8 of the 10 multiplication-division problems, division preceded multiplication, and on problems in which the pattern held, the correct answer was generally the larger number.

The instructional treatment was quite limited in duration; we would prefer to have a higher percentage of students mastering the first test, and we would also prefer that instruction and testing be completed within a week.

Suggestions for Further Research

There are several possible extensions of our study. A retention study would be appropriate to determine the extent to which mastery of both the pattern and its non-examples were retained. Also interviews of subjects might be appropriate to gain further insight into the misapplication of the pattern. The six people who clearly misapplied the pattern are potential candidates for an interview as well as those nine individuals who mastered the mixed sign problems without demonstrating mastery on Test 1.

In our study, we have attempted to determine whether an unintended outcome existed after a particular instructional treatment occurred. Our study indicates that widespread misapplication of the pattern did not occur; perhaps research might indicate that for some other topics it is not necessary to emphasize non-examples during the instruction.

With respect do "doing-undoing" in general, it is suggested that research be done to determine the feasibility of approaching cancellation of fractions from the viewpoint of examples and non-examples of "doing-undoing." Perhaps if students could master the pattern and its non-examples with mixed signs, a more meaningful approach to cancellation could be presented. After learning that $\frac{a \times b}{b}$ can be written as $(a \times b) \div b$ and that the pattern applies in this example, the student might be led to write $\frac{a + b}{b}$ as $(a + b) \div b$ and recognize it as a non-example of the pattern, thus preventing incorrect cancellation from being attempted.

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Appendix 2A

Test 1 and Test 2

NAME: ____________________

SAMPLE PROBLEMS

Circle the correct answer.

A. 27 ÷ 1 = N [✓ N = 27] N = 1 N = Some other number

B. 142 ÷ 2 = N. N =142 N = 2 [✓ N = Some other number]

C. 195 × 0 = N. N =195 [✓ N = 0] N = Some other number

STOP

Don't turn to the next page yet.

Test 1

1. (936 - 527) + 527 = N.   N = 936   N = 527   N = Some other number

2. (86 x 37) ÷ 37 = N.   N = 86   N = 37   N = Some other number

3. 1428 ÷ 51 x 51 = N.   N = 51   N = 3766   N = Some other number

4. 683 + 497 - 497 = N.   N = 497   N = 683   N = Some other number

5. (877 + 539) - 539 = N.   N = 877   N = 539   N = Some other number

6. (928 + 673) - 673 = N.   N = 673   N = 928   N = Some other number

7. 1365 ÷ 21 x 21 = N.   N = 21   N = 1365   N = Some other number

8. (813 + 478) - 478 = N.   N = 478   N = 652   N = Some other number

9. (5162 ÷ 58) x 58 = N.   N = 5162   N = 58   N = Some other number

10. 1508 ÷ 68 x 68 = N.   N = 1508   N = 68   N = Some other number

Test 1

(continued)

11. 663 - 489 + 489 = N. N = 489 N = 751 N = Some other number

12. (49 x 57) ÷ 57 = N. N = 49 N = 57 N = Some other number

13. 3102 ÷ 47 x 47 = N. N = 47 N = 3102 N = Some other number

14. 853 - 264 + 264 = N. N = 853 N = 264 N = Some other number

15. (1674 ÷ 93) x 93 = N. N = 93 N = 1674 N = Some other number

16. 921 - 556 + 556 = N. N = 556 N = 921 N = Some other number

17. (752 - 591) + 591 = N. N = 591 N = 752 N = Some other number

18. 546 ÷ 39 x 39 = N. N = 546 N = 39 N = Some other number

19. (1241 ÷ 73) x 73 = N. N = 73 N = 3568 N = Some other number

20. 789 + 476 - 476 = N. N = 789 N = 476 N = Some other number

Test 2

1. 1519 ÷ (49 x 49) = N. N = 1519 N = 49 N = Some other number

2. (863 + 48) ÷ 48 = N. N = 863 N = 48 N = Some other number

3. 41 x 73 ÷ 41 = N. N= 73 N = 41 N = Some other number

4. (953 - 628) + 628 = N. N = 953 N = 628 N = Some other number

5. (2226 ÷ 42) - 42 = N. N = 2226 N = 42 N = Some other number

6. 477 + 851 - 477 = N. N = 851 N = 477 N = Some other number

7. (58 x 39) + 39 = N. N = 58 N = 39 N = Some other number

8. 93 - 47 x 47 = N. N = 47 N = 93 N = Some other number

9. 1554 ÷ 37 x 37 = N. N = 37 N = 1554 N= Some other number

10. 762 - (419 + 419) = N. N = 419 N = 762 N = Some other number

Test 2

(continued)

11. 412 - 89 ÷ 89 = N. N = 89 N = 412 N = Some other number

12. (586 + 973) - 586 = N. N = 586 N = 973 N = Some other number

13. 2997 ÷ (37 x 37) = N. N = 37 N = 2997 N = Some other number

14. 1512 ÷ 63 + 63 = N. N = 63 N = 1512 N = Some other number

15. 867 + 491 - 491 = N. N = 491 N = 867 N = Some other number

16. (82 x 37) - 37 = N. N = 82 N = 37 N = Some other number

17. (27 x 86) ÷ 27 = N. N = 27 N = 86 N = Some other number

18. (63 x 26) ÷ 26 = N. N = 63 N = 26 N = Some other number

19. 258 + 79 x 79 = N. N = 79 N = 258 N = Some other number

20. 871 - (557 + 557) = N. N = 871 N = 557 N = Some other number

III

INTRODUCING STUDENTS TO REVERSE POLISH NOTATION (RPN)-BASED CALCULATORS

by

Glendon W. Blume and Charles E. Mitchell

Since the hand-held calculator (HHC) has exploded upon the educational scene, many researchers are turning their attention to its uses and implications for classroom instruction. In this rush to study the HHC, it is unfortunate that many implicit assumptions have been made. One example of this is the recommendation that for grades K through 8 we should use an HHC with a simple algebraic logic system with no operational hierarchy (AgL). Some authors such as Birtwistle (1974), Smith (1975), Jesson and Kurley (1975), and Caravella (1976) support their claims with hypotheses, but others ask us to proceed on faith.

Calculator use depends upon the calculator's logic system, or language. Although there are four distinct languages available, only Stultz (1975) and Bell (1976) have suggested using a language other than AgL. Only Bell has suggested exploring the use of an HHC with a Reverse Polish Notation (RPN) logic system in grades K through 8, but as yet, very little has been reported to support the feasibility of its use. Bell does base his recommendations for usage of an RPN calculator on classroom experiences. While his suggestion for using an RPN calculator is considered a good one, we reject several of his reasons. In particular, use of RPN does not necessarily exclude emphasis on topics requiring the use of parentheses, rearrangement of calculations, or recording of intermediate results.

It is the purpose of this study to consider the feasibility of using an HHC equipped with an RPN logic system with middle school students. Problems are taken from the domain of symbolic exemplars emphasizing order of operations, such as a * (b # c) = N, where * and # are elements of {+, -, x, ÷}. Further, we will explore the possibility and desirability of using various flowcharts and recording sheets in conjunction with calculator-based instruction (see Weaver, 1976). Since the students in the sample were familiar with AgL calculators, we hope to acquire some idea of the children's ability to adapt themselves to the RPN language.

It is not the purpose of this paper to rationalize the use of the HHC in the elementary school. Rather we will proceed on the assumption that calculators can be used to enhance learning in some cases.

Method

An intact class of 28 heterogeneously grouped seventh graders from a Madison, Wisconsin middle school formed the sample for the study. These students had experienced instruction with various algebraic logic calculators for one-half hour per week during most of the school year. The 28 students generally worked in pairs, sharing 16 Hewlett-Packard HP-21 calculators which were provided by the experimenters. One student was known to have access to a calculator with an RPN logic system at home; it is assumed that the others had access to RPN calculators only during classtime. The instruction with the HP-21 calculators lasted for 5 weeks with students continuing the one-half hour per week routine.

The authors were responsible for instruction during the 5-day treatment and evaluation. The two regular classroom teachers participated both as observers and by helping students during individual work. Discussion and demonstration were usually done by one person, although there were several times when it was advantageous to have the two authors jointly involved in presenting the lesson. Materials used in the study were developed by the authors.

Day 1

Calculators were distributed to the children. They were told that they were receiving a type of calculator that was basically different from any that they had previously used. No initial period of independent familiarization (or play) was given to the children. Children had the calculators for only 15 minutes during this initial period.

The instruction concerned procedures for simple addition problems such as 8 + 7 = N. The differences between procedures for these simple addition exercises on an RPN calculator and an Agl calculator were emphasized. The keystroke sequences $8\boxed{+}7\boxed{=}$, and $8\boxed{\text{ENTER}\uparrow}7\boxed{+}$ were compared, but the purpose of the $\boxed{\text{ENTER}\uparrow}$ key was not completely explained. Students were simply told that the $\boxed{\text{ENTER}\uparrow}$ key directed the calculator to remember or save the displayed number. The existence of the 4-level operational stack was not mentioned. A detailed description of specific exercises used during each day's instruction is given in Appendix 3A.

Day 2

Procedures for simple addition were reviewed with the students. They also received instruction concerning the other three basic operations. Only one example, such as 12 ÷ 3 = N, was worked for each operation.

It was believed that discussion of the structure and operation of the stack would serve to clarify the way in which numbers and operations are keystroked in an RPN calculator. Consequently, the concept of the 4-level operational stack was introduced early in the second day's instruction.

In discussing the functioning of the stack, three levels of discourse were: (a) talk about what the calculator does, i.e., it saves the number, it needs to remember or store two numbers before a (binary) operation is performed, etc.; (b) include the first, along with some individual method for recording what is stored in the calculator's operational stack. This record would describe the operations performed as a sequence of discrete pictures of the contents of the stack; (c) include the characterizations provided by the first two levels along with an attempt to capture the dynamic nature of the situation, i.e., the movement or the copying and storing of numbers by the process of pushing the stack down (or up).

Since the third level was chosen as the desired mode for discussion of the functioning of the stack, an individual recording sheet (see Figure 3.1) was developed and a concrete model of the stack was constructed for use in demonstrations. The model of the stack illustrated the push-down (up) nature of the stack by utilizing a cardboard slide to which were affixed four slate squares suitable for recording the contents of the stack's four registers. The slide operated on a wooden frame, the background of which displayed permanently stored zeros. The X, Y, Z, and T lables were affixed at the side of the register areas.

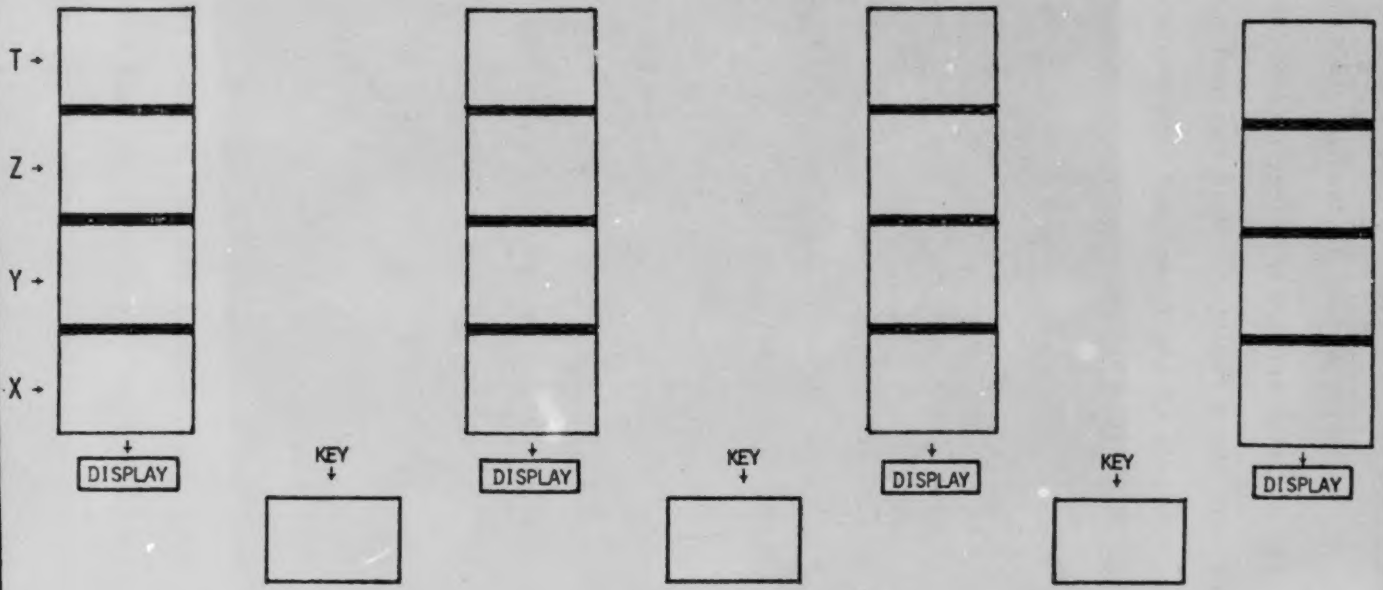
T →
Z →
Y →
X →
DISPLAY
KEY
DISPLAY
KEY
DISPLAY
KEY
DISPLAY

Figure 3.1. Stack recording sheet.

When a number was keyed into the calculator, the slide's first slate square was moved up to the X-position and the appropriate number was written on it. When $\boxed{\text{ENTER}\uparrow}$ was pressed on the calculator, the slide was raised one position and the original number (now in the Y-register) was copied into the X-register. When the next number was keyed in, the previous contents of the X-register were erased, and the new entry was written in. Following the keystroking of an operation, the contents of the X- and Y-registers were erased, the slide was lowered one position exposing the 0 in the Y-register, and the result was written into the X-register. The Z- and T-registers were not used at this time except in mentioning to the students that the calculator placed a zero in all levels of the stack until the user changed them. It was explained that zeros reappeared as the stack "dropped." (This description is not accurate, however, in the case in which the user pushes a non-zero value into the T-register; in this case the value from the T-register is duplicated in the Z-, Y-, and X-registers as the stack falls. Since it was determined that none of the exercises used with the students would require use of the T-register, the less complete, but simpler description of the falling of the stack was chosen for use in this situation.)

The stack recording sheet was used throughout this day's work. Although students were working in pairs and at most one of the two students was using a calculator, each member of the group completed the stack recording sheet after each keystroke. On each example, the concrete model was used in conjunction with the recording sheets;

the stack recording sheet displayed the sequence of changes in the contents of the stack, while with the model, students always had available a visual representation of the current contents of the stack.

When performing an operation such as a * b = N, the students were instructed to ask themselves after each keystroke whether the calculator had enough information to perform the desired operation. Although the term "binary operation" was not mentioned, it was emphasized that the calculator needed to contain both numbers, a and b, before it was possible to indicate and perform the desired operation.

Day 3

Instruction consisted of an initial review of previous work followed by a short period when students could independently work their own exercises. During this period, students were instructed to use the DSP , 0 keystroke sequence so that answers were displayed in integer rather than decimal form. The X-register, Y-register interchange key was then introduced because it was believed that the students would thereby have a means of examining the contents of the stack in their own calculators. Following this, one chaining problem with three numbers and two operations was solved (23 x 79 + 41 = N).

A flowchart format was used for recording results on each of the third day's exercises. As an alternative to the previously used stack recording sheet, this format provided the students with an opportunity to focus on the order in which numbers and operations were keystriked rather than on the machine's internal configuration (stack) after each keystroke. However, the concrete model was again operated by one of the authors and used so that the students always had available for reference purposes a representation of the current contents of the stack. It was also thought that the vertical format of the flowcharts would help the students to recall or visualize the contents of the stack.

A simple linear flowchart was used for problems such as a * b = N (see Figures 3.2a and 3.2b) and a branched flowchart was used for the chaining problem a * b # c = N (see Figure 3.3a). Students were reminded that it was necessary for them to mathematically interpret the chaining problem using the standard order of operations prior to encountering the "Operate?" frame in the flowchart.

Since the students had been introduced to the topic of order of operations earlier in the year, and in light of the half-hour time constraint, only a cursory review of order of operations was conducted prior to work with the chaining problem. The standard order of operations and several examples were discussed relative to order of operations.

Day 4

Instruction included nine chaining examples of the general type a * b # c = N involving three numbers and two operations (see Appendix 3A). The first three examples were worked and discussed as a group. The concrete model of the stack was again used along with a modified version of the previously used branched flowchart (see Figure 3.3b). These three problems included all four basic operations, problems with and without parentheses, and different problems to address each branch of the flowchart.

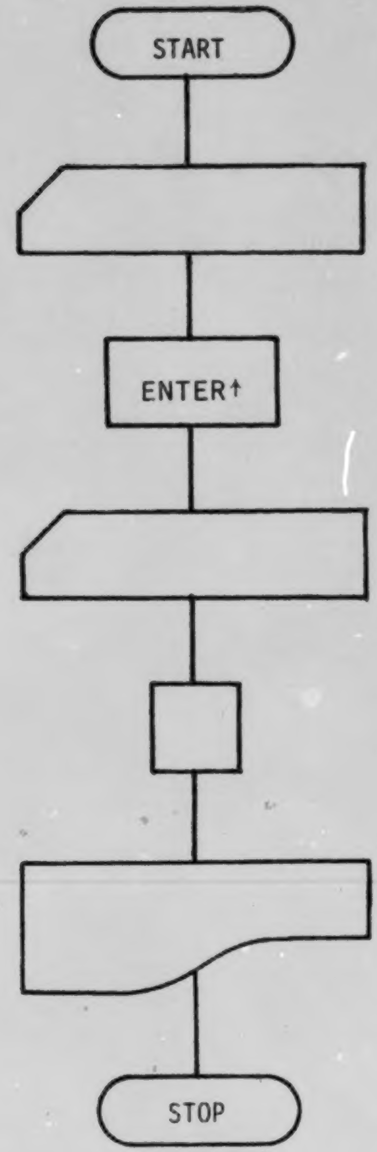
START
ENTER↑
STOP

Figure 3.2a. Linear Flowchart for a * b = N.

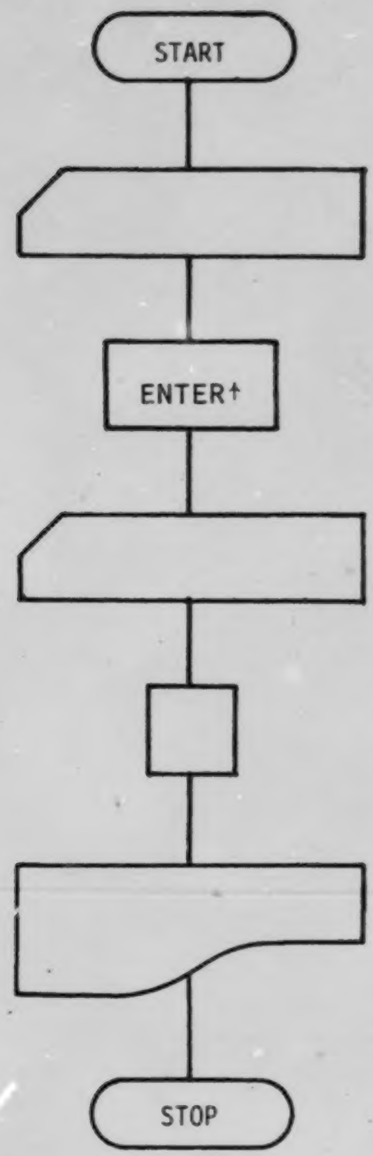
START
ENTER†
STOP

Figure 3.2b. Student response to the linear flowchart.

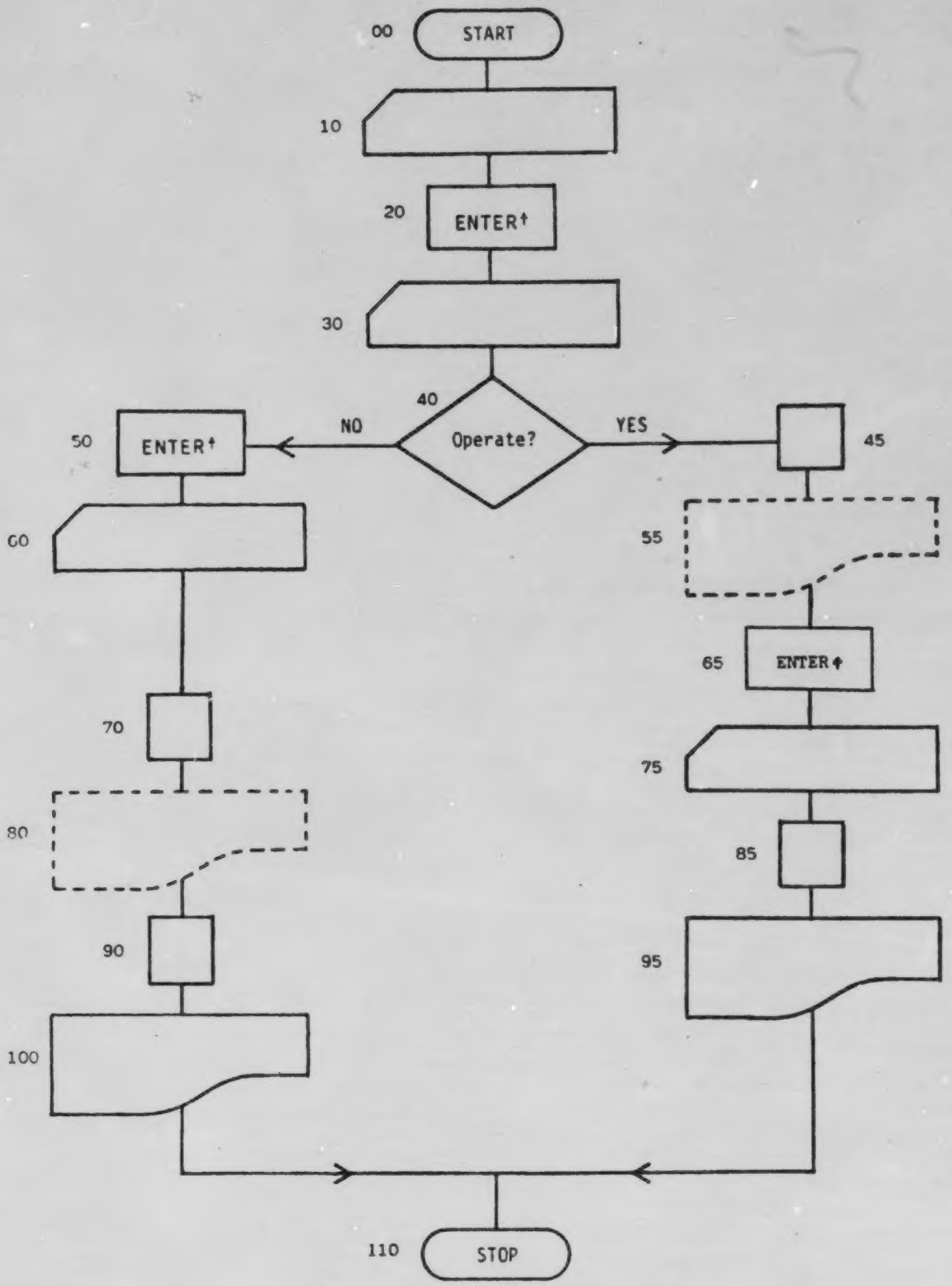
00
START
10
20
ENTER↑
30
40
Operate?
NO
YES
50
ENTER↑
45
60
55
65
ENTER↑
70
75
80
85
90
95
100
110
STOP

Figure 3.3a. Branched flowchart for a * b # c = N.

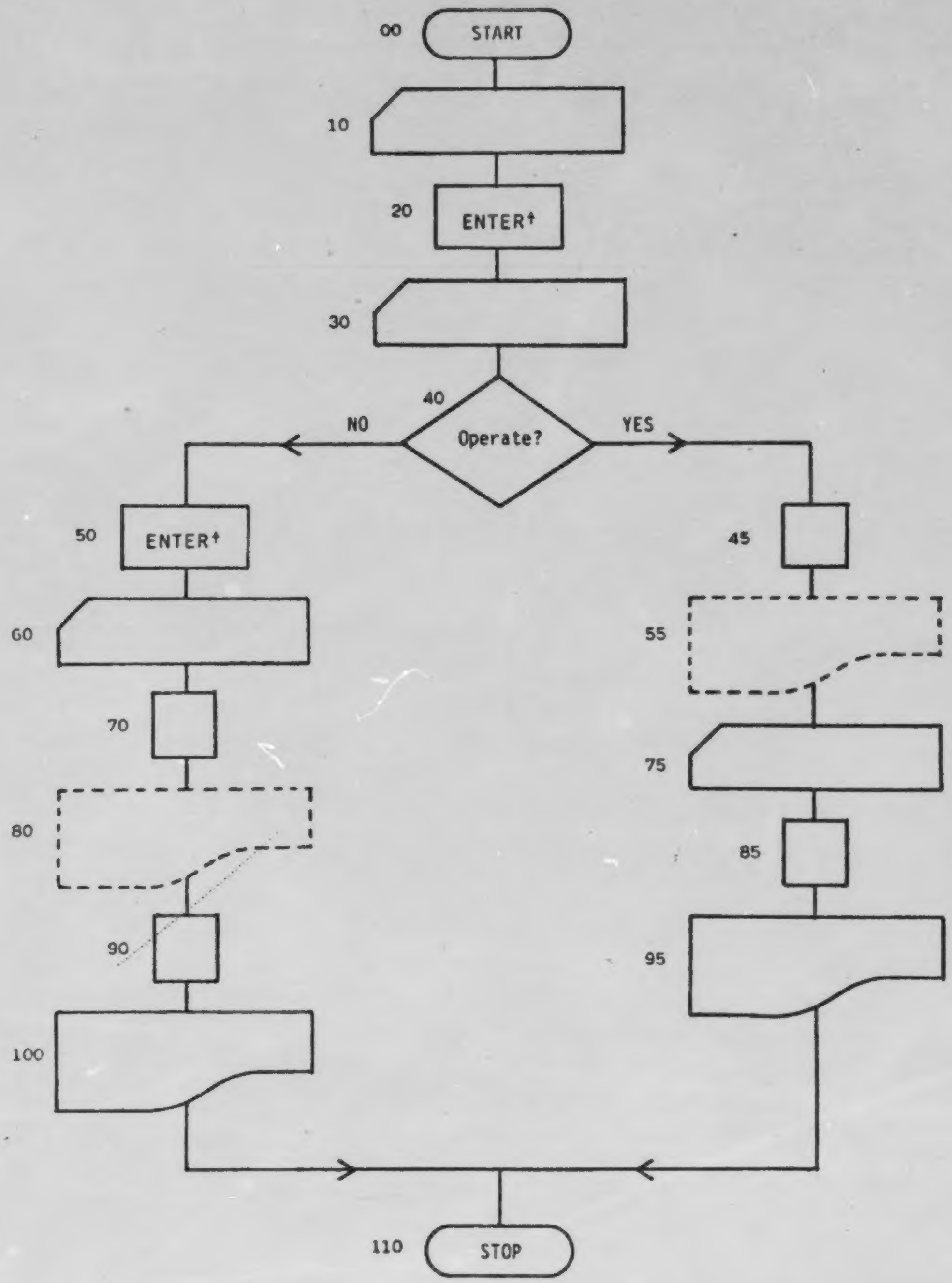
00
START
10
20
ENTER†
30
40
Operate?
NO
YES
50
ENTER†
45
60
55
70
75
80
85
90
95
100
110
STOP

Figure 3.3b. Modified branched flowchart for a * b # c = N.

The remaining six examples were worked by the students in pairs; on one problem, the first student used the calculator while the second recorded keystrokes and results on the flowchart. On the next problem, the students exchanged roles. For these problems, the concrete model was used only for clarification of individual questions.

Day 5

On the final day, the authors administered a 6-item instrument to evaluate pupils' performance on the chaining problems involving three numbers and two operations (see Figures 3.4a and 3.4b). Because it was necessary for most students to share a calculator, the class was randomly divided into two groups; one group worked three problems with the calculator but without the flowcharts (hereafter called the Calculator problems), while the second group worked the other three problems (hereafter called the Flowchart problems) without the calculator by completing the appropriate entries (keystrokes, but not output) in the flowcharts (see Figure 3.3b). When students in either group completed the three problems, their work was collected. They either turned in or received a calculator, and were then given the remaining three problems. It was suggested that students devote approximately equal portions of the 25-minute testing period to the Flowchart problems and to the Calculator problems.

Findings

Of the 28 students who were in the class, only 27 participated in the examination. Of those 27, 11 were unable to complete some portion of the test due to time constraints. Approximately 85% of the questions were completed. (See Appendix 3B for raw response data on each subject.)

For each of the following problems use an HP-21 calculator to find the intermediate and final results. Record your answers in the blank spaces after each problem.

1. (25.97 + 57.78) ÷ 13.4 = n

Intermediate result: ________________

Final result: ________________

2. 39.5 x (15.3 - 6.74) = n

Intermediate result: ________________

Final result: ________________

3. 83.3 - 54.4 ÷ 4.25 = n

Intermediate result: ________________

Final result: ________________

Figure 3.4a. Calculator problems from the 6-item instrument.

For each of the following problems write a number or an operational symbol in each appropriate box checked in red on one of the attached flowchart forms to show how you would work that problem using an HP-21 calculator.

1. (987 - 456) x 321 = n

2. 642 + 5031 ÷ 789 = n

3. 6764 ÷ 89 - 53 = n

Figure 3.4b. Flowchart problems from the 6-item instrument.

Eliminating from consideration those test questions which were not completed, there is very little difference between the performance of the two groups. Therefore, findings will be discussed for the two groups as a whole, and data will be reported only on those questions that were completed. Responses on each question are summarized separately for the two groups in Table 3.1. Table 3.2 summarizes combined responses for the two groups. Table 3.3 presents the data from Table 3.2 for only those items which were completed by the students.

The mean correct response over all students for problems which were completed was 3.73 on a scale of 6, or 62% correct. As can be seen from Table 3.3, results for the Calculator problems were remarkably consistent with those for the Flowchart problems. In both cases, low percentages correct were associated with problems which required knowledge of the mathematical convention concerning order of operations.

Of the Calculator problems involving parentheses (problems 1 and 2), 81% were worked correctly. The few errors that were made involved ignoring parentheses or keying in $\boxed{+}$ instead of $\boxed{\div}$. For the third Calculator problem, which had no parentheses and required the subject to use the order of operations hierarchy to divide before subtracting, only 20% of the subjects responded correctly. The "subtracting before dividing" mistake was made by 40% of the subjects. Less than 7% of the errors were attributed to misuse of the technical features of the RPN language. An example of such would be "incorrectly sequencing the order in which data were entered into the calculator."

For two Flowchart problems (problems 1 and 3), one which involved parentheses and one in which the natural left-to-right order of the

Table 3.1

Results of Responses by Groups to the 6-item Evaluation Instrument

Group 1 (13 Ss)	Calculator Problems			Flowchart Problems		
(Calculator problems first)	1	2	3	1	2	3
Percent Correct	77	85	23	85	23	61
Percent Incorrect	23	15	77	15	62	8
Percent Not Completed	0	0	0	0	15	31

Group 2 (14 Ss)	Calculator Problems			Flowchart Problems		
(Flowchart problems first)	1	2	3	1	2	3
Percent Correct	43	50	7	93	14	79
Percent Incorrect	14	7	43	7	86	21
Percent Not Completed	43	43	50	0	0	0

Table 3.2

Summary of Responses to the 6-item Evaluation Instrument

(Including Non-completed Items)

Combined Groups (27 Ss)	Calculator Problems			Flowchart Problems		
	1	2	3	1	2	3
Percent Correct	59	67	15	89	19	70
Percent Incorrect	19	11	59	11	74	15
Percent Not Completed	22	22	26	0	7	15

Table 3.3

Summary of Responses to the 6 Items

(Ignoring Non-completed Items)

	Calculator Problems			Flowchart Problems		
	1	2	3	1	2	3
Percent Correct	76	86	20	89	20	82
Percent Incorrect	24	14	80	11	80	18

problem followed the order of operations hierarchy, 86% of the subjects successfully answered the questions. Again, less than 7% of the errors could be attributed to misuse of the RPN language. For the second Flowchart problem, in which the natural left-to-right order of the problem differed from the interpretation based on the order of operations hierarchy, 20% of the students responded correctly. Failure to observe the order of operations hierarchy resulted in incorrect response for 61% of the students. Less than 7% of the errors could be attributed to misuse of the RPN language.

Interpretations

Given these findings, it seems reasonable that calculators equipped with an RPN logic system can be successfully operated by a heterogeneous group of seventh graders. As noted earlier, the difficulties encountered by the students were attributable to apparent misapplication of the order of operations hierarchy rather than to difficulty with the technical features of the RPN logic system.

No interference with the use of the RPN logic system was detected from previous usage of Litronix calculators equipped with a simple algebraic hierarchy. The absence of an $\boxed{=}$ key simply forces students to "think RPN."

It is also interesting to note that students were not distracted by the advanced capabilities or unfamiliar notation of the HP-21. This calculator has been recommended for use at the secondary school level by the National Council of Teachers of Mathematics (NCTM, 1977), but it has not been recommended for use at lower levels.

Although the following hypothesis can not be supported or rejected by the data, the use of the concrete model and the record sheet clarified the concept of the 4-level stack. Based on past experiences, the concept of the stack and its various functions have been difficult for beginners to comprehend. The children in the sample asked very few questions about the stack and had few problems performing some basic stack manipulations.

Our experience indicates that there should be one calculator for each child. Sharing calculators increases the chances for distractions such as students talking among themselves about topics other than mathematics. Sharing also reduces the amount of time that a student directly interacts with the calculator.

Initially, children should be allowed a few minutes to manipulate the calculators without the direction of instruction. This is desirable whether or not children have had previous experience with calculators. All that may be necessary for this initial play period is that children by shown how to turn on the calculators.

Limitations

It is unfortunate that so many of the errors that were made involved order of operations. Since no pre-test was given, one can not conclude that use of the hand-held calculator (with the RPN logic system) facilitated or interfered with students' learning of order of operations. The students had received instruction in this area earlier in the year and apparently the short review given by the authors was not sufficient for most students to master this material.

The duration of the study was not of sufficient length to provide coverage of the material that the authors deemed desirable. The students actually worked very few examples. This situation was made worse by having to share calculators.

The testing instrument used in the study also had certain limitations. Although there were no real differences in performance between problems which were done only on the calculator and those done only with the flowchart, it would be worthwhile to also obtain results on problems that required the students to use the calculator in conjunction with the flow charts. Also, no problems included basic exercises such as a * b = N.

Future Considerations

Although the stack was introduced in this study, it may be desirable to consider usage of RPN calculators without introduction of the stack, especially with younger children. It may also be interesting to compare instruction with flowcharts and recording sheets to settings without them.

The children in the sample did have some difficulty with order of operations material. Perhaps they would have less difficulty if they were taught the order of operations hierarchy while using a calculator. On the other hand, the calculator may possess the potential for interfering with previous knowledge of order of operations.

This investigation was exploratory in nature, but valuable insight has been gained into the uses of RPN logic and related problems. Hopefully, more carefully controlled future studies will advance our knowledge of this area. It seems feasible to use RPN calculators with middle school, if not elementary school students, and it is important that researchers begin addressing the question of the calculator logic system and its influence on instructional considerations.

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Appendix 3A

Description of Items Used During Instruction

Day 1:	8 + 7 = N	
Day 2:	8 + 7 = N 8 - 3 = N 7 x 9 = N 12 + 3 = N	Stack recording sheets were used for these four examples.
Day 3:	80 + 20 = N	Flowcharts were used for recording results on problems other than those chosen by the students.
	80 * 20 = N	Where * was either -, x, or ÷ as chosen by the students.
		Several problems similar to the preceding one, again chosen by the student.
	99 - 22 = N	Included here was the use of the $\boxed{x \rightleftarrows y}$ and $\boxed{\text{DSP}}$ keys.
	23 x 79 + 41 = N	
Day 4:	29 x 32 + 67 = N 831 - 17 x 26 = N 868 + (136 + 81) = N	These three items were worked by the class as a group with the stack model; flowcharts were used for recording results.
	289 + 884 + 68 = N 882 + 21 - 27 = N (29 + 19) x 23 = N 726 - 418 - 131 = N 271 - 858 + 66 = N 29 x 736 - 694 = N	These six items were worked individually without the stack model; flowcharts were used for recording results.
Day 5:		The evaluation instrument was administered. (See Figures 3.4a and 3.4b).

Appendix 3B

Responses by Subjects on Each Item of the Day 5 Evaluation Instrument

---

Key:

1 = correct response
0 = incorrect response
n = no response

Group 1 (Calculator problems first)	Subject	Calculator Problems			Flowchart Problems		
		1	2	3	1	2	3
	1	0	0	0	1	0	1
	2	0	1	0	1	0	1
	3*	n	n	n	n	n	n
	4	1	1	1	1	1	1
	5	1	1	0	1	0	1
	6	1	1	1	1	1	1
	7	1	1	0	1	0	1
	8	1	1	0	n	n	n
	9	1	1	0	1	0	0
	10	0	0	0	1	0	0
	11	1	0	0	1	0	1
	12	1	1	0	1	0	n
	13	1	1	0	n	n	n
	14	1	1	1	1	1	1

Group 2 (Flowchart problems first)	Subject	Calculator Problems			Flowchart Problems		
		1	2	3	1	2	3
	1	0	1	1	0	1	1
	2	1	1	0	1	0	1
	3	1	1	0	1	0	1
	4	1	0	0	1	0	1
	5	1	1	n	1	0	1
	6	1	1	0	1	0	1
	7	0	1	0	1	0	1
	8	1	1	0	1	0	0
	9	n	n	n	1	0	1
	10	n	n	n	1	0	1
	11	0	n	n	1	0	1
	12	n	n	n	1	0	0
	13	n	n	n	1	0	0
	14	n	n	n	1	1	1

*This subject refused to do any work and was not included in the summary of responses in Table 1.

IV

STUDENT PERFORMANCE ON A CALCULATOR-BASED "WHAT'S MY RULE?" TASK

by

J. Fred Weaver

137

For one of the instructional periods in November 1977 the EduCALC 29CGD was programmed to use a linear function of the form

$$x \rightarrow 2x - 3, \quad (x \text{ an integer } \nless 2)$$

to generate recursively, given a particular a_1, subsequent terms of the sequence

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots$$

$$\begin{aligned} \text{i.e.,}\quad & a_1 \rightarrow 2a_1 - 3 = a_2, \\ & a_2 \rightarrow 2a_2 - 3 = a_3, \\ & a_3 \rightarrow 2a_3 - 3 = a_4, \\ & \ldots\ldots\ldots\ldots, \\ & a_n \rightarrow 2a_n - 3 = a_{n+1}. \end{aligned}$$

In particular, with an input of 4, the sequence

4, 5, 7, 11, 19, 35, 67, ___

was generated progressively in such a way that each student was to predict and write certain sequence terms before they were displayed by the calculator:

4, 5, 7, $\underline{?}$

4, 5, 7, 11, $\underline{?}$

4, 5, 7, 11, 19, $\underline{?}$

4, 5, 7, 11, 19, 35, $\underline{?}$

4, 5, 7, 11, 19, 35, 67, $\underline{?}$

This task was set in the form of a six-page booklet, reproduced following this page, which was used in the following way:

Page 1. Given the input of 4, the calculator generated as output the next two terms of the sequence: 5 and 7.

Page 2. The student "guessed" the next term of the sequence and wrote that guess in box D.

Page 3. The calculator-generated output for box D was displayed, and then recorded by the student in box D. The student then "guessed" the next term of the sequence and recorded that guess in box E.

Page 4. The calculator-generated output for box E was displayed, and then recorded by the student in box E. The student then "guessed" the next term of the sequence and recorded that guess in box F.

Page 5. The calculator-generated output for box F was displayed, and then recorded by the student in box F. The student then "guessed" the next term of the sequence and recorded that guess in box G.

Page 6. The calculator-generated output for box G was displayed, and then recorded by the student in box G. The student then "guessed" the next term of the sequence and recorded that guess in box H.

This task, with preliminary oral instructions, was group-administered to the 40 students from the class who were present that day. Results are summarized in Table 4.1.

Informal conversations with students revealed that two prediction rules were found to fit the calculator-generated sequence. Each rule was phrased in a variety of ways with varying degrees of grammatical and mathematical precison, and is symbolized as follows:

(1) $a_n \rightarrow 2a_n - 3$ (the function actually programmed into the calculator);

(2) $a_n \rightarrow a_n + 2(a_n - a_{n-1})$, $(n > 1)$.

When the students were then collectively asked to predict sequence terms beyond $a_8 = 131$ before each was calculator-displayed (but with no written record made), students using rule (1) were able to apply that rule much more readily than were students who attempted to use rule (2). No attempt was made to ascertain why the seemingly different rules generated the same predicted terms.

Table 4.1

Students' Predictions of Sequence Terms

Subject	Task-booklet page				
	2	3	4	5	6
	Response box				
	D	E	F	G	H
	Term of sequence				
	4th	5th	6th	7th	8th
	Value of term				
	11	19	35	67	131
	Subject response				
1.	10	16	35	42	144
2.	10	16	24	NR	131
3.	10	17	24	43	131
4.	10	16	28	67	131
5.	10	15	28	67	131
6.	8	16	29	67	131
7.	10	17	35	67	131
8.	10	19	35	67	131
9.	10	19	35	67	131
10.	10	17	26	67	131
11.	10	19	35	67	131
12.	10	19	35	67	131
13.	10	17	28	67	131
14.	10	17	21	45	130
15.	8	19	35	67	201
16.	10	13	26	67	131
17.	10	15	28	67	102
18.	10	13	25	47	71
19.	11	19	35	67	131
20.	10	15	35	67	131
21.	10	16	24	67	133
22.	10	16	27	43	112
23.	10	17	25	48	131
24.	10	16	35	42	132
25.	10	17	23	56	130
26.	10	19	35	67	131
27.	10	16	23	70	134
28.	10	22	26	71	135
29.	10	16	31	67	121
30.	10	16	30	51	89
31.	11	19	35	67	131
32.	10	13	35	67	131
33.	10	19	35	67	131
34.	10	17	28	59	124
35.	11	19	35	67	131
36.	10	19	35	67	131
37.	10	17	30	54	80
38.	10	19	35	67	131
39.	10	19	35	67	131
40.	10	19	35	67	131
#	3	14	19	26	25

Notes: NR: No Response
#: Number of subjects responding with the recursively determined "value of term"

One week later (11-18-77) the same seventh grade students were given the following sheet to work with individually:

A	4	A	6
B	5	B	______
C	7	C	______
D	11	D	______
E	19	E	______
F	35	F	______
G	67		
H	______		

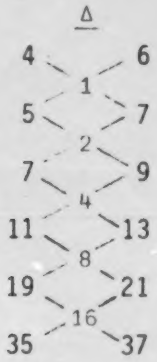
Each student was to recall or reconstruct some rule (not necessarily the one "discovered" by him or her last week) for the sequence at the left and then apply that rule to 6 at the right to generate a new 6-term sequence. It was observed that:

2 students failed to generate all of the terms B through F of a sequence, as requested and expected;

7 students generated sequences of one sort or another that appeared to be confounded by computational error(s) in calculating some term(s);

26 students generated the sequence 6, 9, 15, 27, 51, 99 [very likely using recursive rule (1) identified on preceding page 3]; and

5 students generated the sequence 6, 7, 9, 13, 21, 37 [in all likelihood proceeding in accord with the diagram below by using the pattern of differences (Δ) derived from terms of the left-hand sequence to generate terms of the right-hand sequence].



This observation that two (or more) different rules might "fit" a given sequence but generate different sequences when applied to a different "starting number" (or first term) is not unique to the illustration cited. It has been reported that in one of the CSMP developmental classes (Comprehensive School Mathematics Program, 1977):

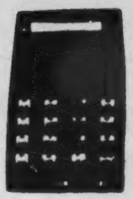
A student wrote this sequence on the board: 5; 8; 14; 26; 50;

S. Your rule is 2 x followed by a -2.

S. That works, but I was adding 3, then adding 6, then adding 12, then adding 24, and so on [p. 180].

If each student had then been asked to apply her or his rule to the number 7, to generate a new sequence, the first student's rule would have led to 7, 12, 22, 42; whereas the other student's rule would have led to 7, 10, 16, 28.

Pilot Sequence Task Used with Grade 7 Subjects



INPUT

A 4

OUTPUT

R/S B 5

R/S C 7

STOP

DON'T
GO ON
YET

INPUT

A 4

OUTPUT

R/S	B	5
R/S	C	7
R/S	D	?

STOP

DON'T
GO ON
YET

INPUT

	A	4

OUTPUT

R/S	B	5
R/S	C	7
R/S	D	
R/S	E	?

STOP

DON'T
GO ON
YET

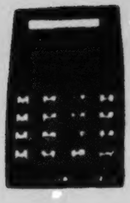
INPUT

A	4

OUTPUT

R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	
R/S	F	?

DON'T
GO ON
YET



INPUT

A 4

OUTPUT

R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
R/S	F	
R/S	G	?

STOP

DON'T
GO ON
YET

INPUT

A 4

OUTPUT

R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
R/S	F	35
R/S	G	
R/S	H	?

STOP

THE END

TABLE 4.2

Levels of Simultaneous/Progressive Presentations of Terms in an Illustrative Sequence Task

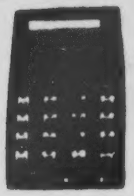
Sequence term		Task											
		S-1		S-2				S-3					
Box	Value	Booklet page											
		1	2	1	2	3	4	1	2	3	4	5	6
A	4	x	x	x	x	x	x	x	x	x	x	x	x
B	5	x	x	x	x	x	x	x	x	x	x	x	x
C	7	x	x	x	x	x	x	x	x	x	x	x	x
D	11	x	x	x	x	x	x		?	$\boxed{x}$	x	x	x
E	19	x	x	x	x	x	x			?	$\boxed{x}$	x	x
F	35	x	x		?	$\boxed{x}$	x				?	$\boxed{x}$	x
G	67	x	x			?	$\boxed{x}$					?	$\boxed{x}$
H	131		?				?						?

Key. x : Value of sequence term is already printed in the task booklet.

? : Subject is to predict and write the sequence term in the task booklet.

$\boxed{x}$: Subject is to write in the task booklet the calculator-generated and calculator-displayed value of the sequence term.

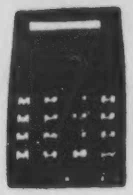
Illustrative Sequence Task S-1



START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
R/S	F	35
R/S	G	67

STOP

DON'T
GO ON
YET



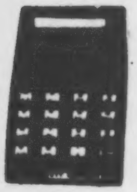
START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
R/S	F	35
R/S	G	67
	H	?

STOP

DON'T
GO ON
YET
OR GO
BACK

WHAT'S THE RULE? REMEMBER IT.

DON'T TELL ANYONE. KEEP IT A SECRET.



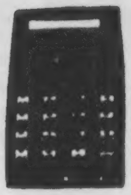
NOW, USE THE RULE.

WRITE YOUR ANSWERS
IN THE BOXES.

START WITH THIS NUMBER:	6	A
WHAT'S THE NEXT NUMBER?		B
WHAT'S THE NEXT NUMBER?		C
WHAT'S THE NEXT NUMBER?		D
WHAT'S THE NEXT NUMBER?		E
WHAT'S THE NEXT NUMBER?		F

STOP
THE END
DON'T
GO BACK

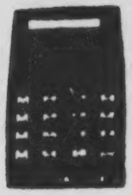
Illustrative Sequence Task S-2



START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19

STOP

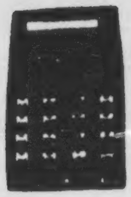
DON'T
GO ON
YET



START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
	F	?

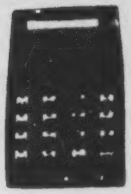
STOP

DON'T
GO ON
YET
OR GO
BACK



START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
R/S	F	
	G	?

STOP
DON'T
GO ON
YET
OR GO
BACK



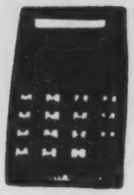
START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
R/S	F	35
R/S	G	
	H	?

STOP

DON'T
GO ON
YET
OR GO
BACK

WHAT'S THE RULE? REMEMBER IT.

DON'T TELL ANYONE. KEEP IT A SECRET.



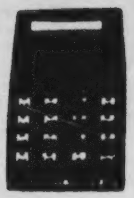
NOW, USE THE RULE.

WRITE YOUR ANSWERS
IN THE BOXES.

START WITH THIS NUMBER:	6	A
WHAT'S THE NEXT NUMBER?		B
WHAT'S THE NEXT NUMBER?		C
WHAT'S THE NEXT NUMBER?		D
WHAT'S THE NEXT NUMBER?		E
WHAT'S THE NEXT NUMBER?		F

STOP
THE END
DON'T
GO BACK

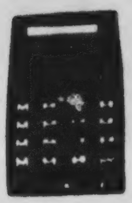
Illustrative Sequence Task S-3



START WITH:	A	4
R/S	B	5
R/S	C	7

STOP

DON'T
GO ON
YET



START WITH: A 4

R/S B 5

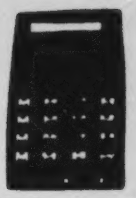
R/S C 7

D ?

STOP

DON'T GO ON YET

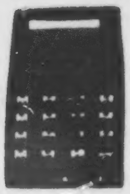
OR GO BACK



START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	
	E	?

STOP

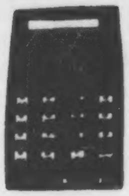
DON'T
GO ON
YET
OR GO
BACK



START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	
	F	?

STOP

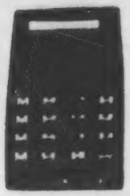
DON'T
GO ON
YET
OR GO
BACK



START WITH:		A	4
	R/S	B	5
	R/S	C	7
	R/S	D	11
	R/S	E	19
	R/S	F	
		G	?

STOP

DON'T
GO ON
YET
OR GO
BACK



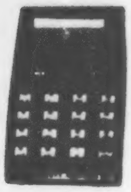
START WITH:	A	4
R/S	B	5
R/S	C	7
R/S	D	11
R/S	E	19
R/S	F	35
R/S	G	
	H	?

STOP

DON'T
GO ON
YET
OR GO
BACK

WHAT'S THE RULE? REMEMBER IT.

DON'T TELL ANYONE. KEEP IT A SECRET.



NOW, USE THE RULE.

WRITE YOUR ANSWERS
IN THE BOXES.

START WITH THIS NUMBER: 6 A

WHAT'S THE NEXT NUMBER? B

WHAT'S THE NEXT NUMBER? C

WHAT'S THE NEXT NUMBER? D

WHAT'S THE NEXT NUMBER? E

WHAT'S THE NEXT NUMBER? F

STOP
THE END
DON'T
GO BACK

Future Work

This pilot work has prompted the writer to develop a more extensive plan to investigate students' spontaneous responses to function-generated, calculator-mediated number-pattern tasks: "spontaneous" in the sense that students will have received only negligible, if any, instruction pertaining to criterion-like tasks so that students will respond on the basis of "open search" rather than "algorithmic thinking" (Avital & Shettelworth, 1968), unless some of the latter develops and emerges during the course of work on a set of criterion tasks.

One of the variables to be included will be illustrated explicitly: the degree or extent to which sequence terms are presented "simultaneously" vs. "progressively." In this connection, three levels of task type have been identified. They are characterized in Table 4.1 and are illustrated in the sample pages following that table.

Ginsburg (1977) contends that "Children must learn not only to calculate, but no perceive, to see. They must learn that mathematics is concerned with important regularities, and that numbers behave in orderly ways that they can predict [p. 160]."

The programmable calculator appears to have considerable potential in investigating some aspects of this kind of learning, and also in facilitating it within a context of direct, explicit instruction.

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V

A MONADIC MODULE--ALIAS, A UNARY UNIT

by

J. Fred Weaver

An abridged verson of the material in this chapter was published in Calculators/Computers Magazine, 1978, 2(4), 29-35.

This module is excerpted from material prepared for ongoing calculator explorations with a class of accelerated seventh-grade students who will study "first-year algebra" in grade 8 rather than in grade 9. The module is predicated upon several premises, and its use assumes certain prior mathematical and calculator experiences.

Some Premises

1. Unary or monadic operations should receive greater explicit attention than usually is the case in school mathematics programs which put such heavy emphasis upon binary or diadic operations (Hess, 1974).

2. Conventional arithmetic or algebraic notation for unary operators very readily may confuse rather than clarify for school learners the *conceptualization* of unary operations. For instance:

$$\log x \qquad \sqrt{x} \text{ or } x^{\frac{1}{2}}$$

$$x! \qquad x^2$$

$$|x| \qquad x^{-1} \text{ or } \frac{1}{x} \text{ or } 1/x$$

$$[x] \qquad e^x$$

As Gilman & Rose (1976) have indicated, "Whatever other merits this mishmash has, consistency certainly isn't one of them, for the symbol which is the functional indicator may appear on the left, the right, both sides, on top, or be in a special position, or be represented by an alphabetical label [p. 57]!"

3. At a pre-algebra level in particular, potential confusion can be minimized and conceptual clarification enhanced by using in connection with all unary functions a consistent *post*-operator or *post*fix notation of the form "$a\boxed{\nabla}$" that in itself suggests the manner in which such functions are *calculator-executed*: first, operand a is entered or otherwise displayed; then a is "acted upon" by the operation symbolized by operator $\boxed{\nabla}$. Using this calculator-based notation or language, it then is sensible to

write statements of the form $a \rightarrow a\boxed{\nabla}$ or $a\boxed{\nabla} = b$.

More specifically, the three unary operations that are the focus of this two-part module may be designated by the following operators:

$\boxed{x^2}$: the *squaring* operator;

$\boxed{\sqrt{\ }}$ or $\boxed{\sqrt{x}}$: the *square root* operator; and

$\boxed{1/x}$: the *reciprocal* or *multiplicative inverse* operator.

Although a "mishmash" of *labels* remains embedded in the preceding symbols, the mishmash of *operator* designation can be eliminated:

Conventional notation	Calculator-based notation
$a \rightarrow a^2$ or $a^2 = b$	$a \rightarrow a\boxed{x^2}$ or $a\boxed{x^2} = b$
$a \rightarrow \sqrt{a}$ or $\sqrt{a} = b$	$a \rightarrow a\boxed{\sqrt{\ }}$ or $a\boxed{\sqrt{\ }} = b$
$a \rightarrow \frac{1}{a}$ or $\frac{1}{a} = b$	$a \rightarrow a\boxed{1/x}$ or $a\boxed{1/x} = b$

Two things should be noted at this point.

First, although the calculator-based notation *suggests* in each instance a keystroke sequence to be used when applying a particular unary operation to some number a, this understandably does not make explicit that a "prefix" or "shift" key must be used on some calculators to select the desired unary operation, and that for $a < 0$ a CHS or +/- key must be used along with digit key(s) to enter a.

Second, mention of the CHS or +/- key may raise the following question: Why is not the *additive inverse* operation included among the unary operations in this module? A user of the module certainly may elect to make that inclusion. But it has been my experience with middle-school students at the pre-algebra level that they have *considerable* difficulty distinguishing between use of the CHS or +/- key in connection with data entry and use of the *same* key to perform the additive - inverse *unary operation* upon a number. This is the same problem that students encounter when ininterpreting an expression such as "-(-5)" in which the same symbol, - , is to be viewed in different ways. For this reason I have bypassed the additive-inverse operation in *this* module.

4. Finally, work with unary operations should emphasize properties and relationships. Several of these provide the basis for this module.

Pre-module Background

It is assumed that:

1. Students have worked with *rational* numbers (negative as well as non-negative), and readily can order such numbers in decimal form and can either *truncate* or *round* decimal representations of rational numbers to a specified number of decimal places.

2. Students' work with calculators has included exemplars of each of the following where a is rational and b may be either "exact" or a rational *approximation* (or even *undefined*):

Input	Function key	Output	Interpretation(s)
a	$\boxed{x^2}$	b	$b = a \times a$
a	$\boxed{\sqrt{\ }}$	b	$b \times b = a$ or $a \div b = b$
a	$\boxed{1/x}$	b	$1 \div a = b$ or $b \times a = 1$

The kind(s) of calculator(s) used by students--prior to the module or for the work of the module--is immaterial as far as "logic" is concerned: it may be algebraic, without or with an operational hierarchy, or it may be RPN (reverse Polish notation). The calculators that I have used with the seventh-grade class are RPN-based: specifically, National Semiconductor's "Mathematician" 4510 (and, for demonstration, etc., EduCALC's 29GD which utilizes a Hewlett-Packard 29C programmable).

The particular relationships and properties to be identified by students may be summarized as follows, where a is a rational number:

Part 1

	#1	#2	#3
Condition	$a \boxed{x^2} = b$	$a \boxed{\sqrt{\ }} = b$	$a \boxed{1/x} = b$
$a > 1$	$b > a$	$b < a$	$b < a$
$a = 1$	$b = a$	$b = a$	$b = a$
$0 < a < 1$	$b < a$	$b > a$	$b > a$
$a = 0$	$b = a$	$b = a$	Undefined
$-1 < a < 0$	$b > a$	Undefined	$b < a$
$-1 = a$			$b = a$
$-1 > a$			$b > a$

Part 2

	#4	#5	#6
Condition	$a \boxed{x^2}\ \boxed{1/x} = c$ $a \boxed{1/x}\ \boxed{x^2} = c'$	$a \boxed{1/x}\ \boxed{\sqrt{\ }} = c$ $a \boxed{\sqrt{\ }}\ \boxed{1/x} = c'$	$a \boxed{\sqrt{\ }}\ \boxed{x^2} = c$ $a \boxed{x^2}\ \boxed{\sqrt{\ }} = c'$
$a > 0$	$c' = c$	$c' = c$	$c' = c = a$
$a = 0$	c & c' undefined	c & c' undefined	
$a < 0$	$c' = c$		c undefined (but not c')

#7. $a \boxed{1/x}\ \boxed{1/x} = a$ for all $a \neq 0$.

Teachers may use the record sheets in a variety of ways. It is intended that these sheets summarize evidence in the form of specific numerical exemplars generated by students (subject to teacher suggestion or guidance as needed to provide an adequate input domain) that can lead either to a discovery of particular relationships/properties or to a support or denial of hypothesized or conjectured relationships/properties.

It is suggested that teachers use their own decision rules regarding the recording of calculator output in truncated or round form. It is important, however, that students be aware of the following fact and its potential implications when working with Part 2 questions: The second of the two unary operations is applied to the digits retained internally by the calculator, which may be more precise than a recorded truncated or rounded value of b or b'. The properties/relationships identified in this paper apply to rational numbers but must be inferred at times from decimal representations that are approximate rather than exact.

The record sheets for questions #4, #5, and #6 provide for first recording the steps of a calculator algorithm, several of which are illustrated in Table 5.1 for the question #4 record. Depending upon the calculators used by students, such algorithms may require that a be input via the keyboard twice rather than once in connection with each exemplar.

Appendix 5A details the user instructions and program that pertain to an EduCALC GD29 that was used to advantage when working with students on this module. Once the calculator has been initialized for a particular question and record sheet, the generation of output is facilitated by having to execute only the R/S key, regardless of the unary operation(s) involved.

It is left to the teacher's discretion to decide the extent to which students should be able to verbalize the relationships and properties emphasized in this module. In Part 2, the teacher may or may not wish to make explicit that:

the operators in #4, #5, and #6 may be commuted;
the operators in #6 are inverses of each other; and
the operator in #7 is its own inverse.

#1. How is $a\boxed{x^2}$ related to a ?

Is it the same for every a ?

a	$a\boxed{x^2} = b$	b ? a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a

#2. How is $a\boxed{\sqrt{}}$ related to a ?
Is it the same for every a ?

a	$a\boxed{\sqrt{}} = b$	b ? a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a

#3. How is $a\ \boxed{1/x}$ related to a ?

Is it the same for every a ?

a	$a\ \boxed{1/x} = b$	b ? a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a
		b a

Calculator Algorithm for Question # 4

Step	Input	Key(s)	Output

#4. How does $a\boxed{1/x}\,\boxed{x^2}$ compare with $a\boxed{x^2}\,\boxed{1/x}$? The same for all a ?

a	$a\boxed{x^2} = b$	$b\boxed{1/x} = c$	$a\boxed{1/x} = b'$	$b'\boxed{x^2} = c'$	c' ? c
					c' c
					c' c
					c' c
					c' c
					c' c
					c' c
					c' c

TABLE 5.1

Calculator Algorithm for Question # 4

Step	Input	Key(s)		Output
1	a	ENT		a
2		F	x^2	b
3		$1/x$		c
4		C↓		a
5		$1/x$		b'
6		F	x^2	c'

The algorithm above is one of three possibilities suggested by students using the RPN "4510" calculators.

The algorithms at the right are the three algorithms, expressed in a generalized form, that apply to questions 4, 5, 6 when RPN "4510" calculators are used.

Step	Input	Key(s)		Output
1	a	ENT		a
2		∇_1		$a \boxed{\nabla_1} = b$
3		∇_2		$b \boxed{\nabla_2} = c$
4		C↓		a
5		∇_2		$a \boxed{\nabla_2} = b'$
6		∇_1		$b \boxed{\nabla_1} = c'$

OR

1	a	ENT		a
2		∇_1		b
3		∇_2		c
4		$x \leftrightarrow y$		a
5		∇_2		b'
6		∇_1		c'

OR

1	a	MS		a
2		∇_1		b
3		∇_2		c
4		MR		a
5		∇_2		b'
6		∇_1		c'

Calculator Algorithm for Question # 5

Step	Input	Key(s)	Output
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	

#5. How does $a\,\boxed{\sqrt{\ }}\,\boxed{1/x}$ compare with $a\,\boxed{1/x}\,\boxed{\sqrt{\ }}$? The same for all a ?

a	$a\,\boxed{1/x} = b$	$b\,\boxed{\sqrt{\ }} = c$	$a\,\boxed{\sqrt{\ }} = b'$	$b'\,\boxed{1/x} = c'$	c' ? c
					c' c
					c' c
					c' c
					c' c
					c' c
					c' c
					c' c

Calculator Algorithm for Question # 6

Step	Input	Key(s)	Output
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	

#6. How does $a\boxed{\sqrt{\ }}\ \boxed{x^2}$ compare with a ? The same for all a ?

How does $a\boxed{x^2}\ \boxed{\sqrt{\ }}$ compare with a ? The same for all a ?

a	$a\boxed{\sqrt{\ }} = b$	$b\boxed{x^2} = c$	$a\boxed{x^2} = b'$	$b'\boxed{\sqrt{\ }} = c'$	$c\ ?\ a$ $c'\ ?\ a$
					$c \quad a$ $c' \quad a$
					$c \quad a$ $c' \quad a$
					$c \quad a$ $c' \quad a$
					$c \quad a$ $c' \quad a$
					$c \quad a$ $c' \quad a$
					$c \quad a$ $c' \quad a$
					$c \quad a$ $c' \quad a$

#7. What is $\boxed{\nabla}$ so that $a\,\boxed{1/x}\,\boxed{\nabla} = a$, for $a \neq 0$?

a	$a\,\boxed{1/x} = b$	$b\,\boxed{\nabla} = a$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$
		$b\,\boxed{} =$

References

Gilman, L., & Rose, A. J. APL, an interactive approach (2nd ed. rev. reprint). New York: Wiley, 1976.

Hess, A. L. Unary operations. Mathematics Teacher, 1974, 67, 281-283.

APPENDIX 5A

EduCALC 29GD User Instruction and Part 2 Illustration

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		OUTPUT DATA UNITS
1	Set display for p places (May be changed at any time)		f FIX	p	
	PART 1. Execute steps 2 thru 4.				
2	Initialize for ONE of these: for $\boxed{x^2}$	1	g RTN	R/S	0
	OR for $\boxed{\sqrt{x}}$	2	g RTN	R/S	0
	OR for $\boxed{1/x}$	3	g RTN	R/S	0
3	Iterate this for each a $\boxed{\nabla}$	a	R/S		$a\boxed{\nabla}=b$
4	For new $\boxed{\nabla}$, return to step 2.				
	PART 2. Execute steps 5 thru 7.				
5	Initialize for ONE of these permutations:				
	for $\boxed{x^2}\boxed{1/x}$, $\boxed{1/x}\boxed{x^2}$	4	g RTN	R/S	0
	OR for $\boxed{\sqrt{x}}\boxed{1/x}$, $\boxed{1/x}\boxed{\sqrt{x}}$	8	g RTN	R/S	0
	OR for $\boxed{x^2}\boxed{\sqrt{x}}$, $\boxed{\sqrt{x}}\boxed{x^2}$	9	g RTN	R/S	0
	OR for $\boxed{1/x}\boxed{x^2}$, $\boxed{x^2}\boxed{1/x}$	7	g RTN	R/S	0
	OR for $\boxed{1/x}\boxed{\sqrt{x}}$, $\boxed{\sqrt{x}}\boxed{1/x}$	5	g RTN	R/S	0
	OR for $\boxed{\sqrt{x}}\boxed{x^2}$, $\boxed{x^2}\boxed{\sqrt{x}}$	6	g RTN	R/S	0
6	Iterate for each a & selected permut.:	a	R/S		$a\boxed{\nabla_1}=b$
			R/S		$b\boxed{\nabla_2}=c$
			R/S		$a\boxed{\nabla_2}=b'$
			R/S		$b'\boxed{\nabla_1}=c'$
7	For new permutation, return to step 5				

NOTE: To generate $a\boxed{\nabla}\boxed{\nabla}$......
keystroke a R/S R/S ....

4	g RTN	R/S	0
-5.12	R/S		26.2144
	R/S		0.0381
	R/S		-0.1953
	R/S		0.0381

If display shows Error at any time, push $\boxed{CLx}$ $\boxed{GTO}$ $\boxed{1}$. Then return to step 3 (Part 1) or 6 (Part 2) with a new a.

EduCALC 29GD Program

Key(s)	Line	Code
STO 0	01	23 00
CLx	02	34
GTO 0	03	13 00
g LBL 6	04	15 13 06
STO 1	05	23 01
f $\sqrt{x}$	06	14 63
R/S	07	74
g x^2	08	15 63
R/S	09	74
RCL 1	10	24 01
g x^2	11	15 63
R/S	12	74
GTO 2	13	13 02
g LBL 8	14	15 13 08
STO 1	15	23 01
f $\sqrt{x}$	16	14 63
R/S	17	74
g $1/x$	18	15 74
R/S	19	74
RCL 1	20	24 01
g $1/x$	21	15 74
R/S	22	74
g LBL 2	23	15 13 02
f $\sqrt{x}$	24	14 63
GTO 0	25	13 00
g LBL 5	26	15 13 05
STO 1	27	23 01
g $1/x$	28	15 74
R/S	29	74
f $\sqrt{x}$	30	14 63
R/S	31	74
RCL 1	32	24 01
f $\sqrt{x}$	33	14 63
R/S	34	74
GTO 3	35	13 03
g LBL 7	36	15 13 07
STO 1	37	23 01
g $1/x$	38	15 74
R/S	39	74
g x^2	40	15 63
R/S	41	74
RCL 1	42	24 01
g x^2	43	15 63
R/S	44	74
g LBL 3	45	15 13 03
g $1/x$	46	15 74
GTO 0	47	13 00
g LBL 9	48	15 13 09
STO 1	49	23 01
g x^2	50	15 63
R/S	51	74
f $\sqrt{x}$	52	14 63
R/S	53	74
RCL 1	54	24 01
f $\sqrt{x}$	55	14 63
R/S	56	74
GTO 1	57	13 01
g LBL 4	58	15 13 04
STO 1	59	23 01
g x^2	60	15 63
R/S	61	74
g $1/x$	62	15 74
R/S	63	74
RCL 1	64	24 01
g $1/x$	65	15 74
R/S	66	74
g LBL 1	67	15 13 01
g x^2	68	15 63
g LBL 0	69	15 13 00
R/S	70	74
GTO i	71	13 22

VI

SOME MONADIC/DIADIC COMBOS

by

J. Fred Weaver

An abridged verson of the material in this chapter was published in Calculators/Computers Magazine, 1978 2(5), 79-83.

This unit is an extension of A Monadic Module, Alias a Unary Unit (see Chapter 5). It is assumed that students have worked through that material as background for this unit. Such background will not be repeated here.

The explorations to be made are associated with a set of questions that may be expressed algebraically in the following form:
Under what condition(s) for a and b is it true that $d = c$ when:

# 1. $(a + b)^2 = c$ and $a^2 + b^2 = d$?

# 2. $(a - b)^2 = c$ and $a^2 - b^2 = d$?

# 3. $(a \times b)^2 = c$ and $a^2 \times b^2 = d$?

# 4. $(a \div b)^2 = c$ and $a^2 \div b^2 = d$?

# 5. $\sqrt{a + b} = c$ and $\sqrt{a} + \sqrt{b} = d$?

# 6. $\sqrt{a - b} = c$ and $\sqrt{a} - \sqrt{b} = d$?

# 7. $\sqrt{a \times b} = c$ and $\sqrt{a} \times \sqrt{b} = d$?

# 8. $\sqrt{a \div b} = c$ and $\sqrt{a} \div \sqrt{b} = d$?

# 9. $\frac{1}{a + b} = c$ and $\frac{1}{a} + \frac{1}{b} = d$ [or $(a + b)^{-1} = c$ and $a^{-1} + b^{-1} = d$] ?

#10. $\frac{1}{a - b} = c$ and $\frac{1}{a} - \frac{1}{b} = d$ [or $(a - b)^{-1} = c$ and $a^{-1} - b^{-1} = d$] ?

#11. $\frac{1}{a \times b} = c$ and $\frac{1}{a} \times \frac{1}{b} = d$ [or $(a \times b)^{-1} = c$ and $a^{-1} \times b^{-1} = d$] ?

#12. $\frac{1}{a \div b} = c$ and $\frac{1}{a} \div \frac{1}{b} = d$ [or $(a \div b)^{-1} = c$ and $a^{-1} \div b^{-1} = d$] ?

However, in terms of the calculator-influenced unary-operator notation of Chapter 5, these same questions may be posed in a different way:

Under what condition(s) for a and b is it true that $d = c$ when:

#1. $(a + b)\boxed{x^2} = c$ and $(a\boxed{x^2}) + (b\boxed{x^2}) = d$?
#2. $(a - b)\boxed{x^2} = c$ and $(a\boxed{x^2}) - (b\boxed{x^2}) = d$?
#3. $(a \times b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \times (b\boxed{x^2}) = d$?
#4. $(a \div b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \div (b\boxed{x^2}) = d$?

#5. $(a + b)\boxed{\sqrt{}} = c$ and $(a\boxed{\sqrt{}}) + (b\boxed{\sqrt{}}) = d$?
#6. $(a - b)\boxed{\sqrt{}} = c$ and $(a\boxed{\sqrt{}}) - (b\boxed{\sqrt{}}) = d$?
#7. $(a \times b)\boxed{\sqrt{}} = c$ and $(a\boxed{\sqrt{}}) \times (b\boxed{\sqrt{}}) = d$?
#8. $(a \div b)\boxed{\sqrt{}} = c$ and $(a\boxed{\sqrt{}}) \div (b\boxed{\sqrt{}}) = d$?

#9. $(a + b)\boxed{1/x} = c$ and $(a\boxed{1/x}) + (b\boxed{1/x}) = d$?
#10. $(a - b)\boxed{1/x} = c$ and $(a\boxed{1/x}) - (b\boxed{1/x}) = d$?
#11. $(a \times b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \times (b\boxed{1/x}) = d$?
#12. $(a \div b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \div (b\boxed{1/x}) = d$?

The preceding format is the one used with a class of accelerated seventh-grade students (who will study "first-year algebra" in grade 8 rather than in grade 9) in connection with their calculator explorations.

In contrast to the "mishmash" of conventional algebraic notation, the consistency of the calculator-based notation makes it much more evident to students that all questions are of a common form involving sentences

$$(a \circ b)\boxed{\nabla} = c \quad \text{and} \quad (a\boxed{\nabla}) \circ (b\boxed{\nabla}) = d$$

and that a question of *distributivity*--of ∇ over $\circ$ --is an overriding one throughout the instructional explorations.

Instructional Materials

In addition to a suitable calculator, each student should have the following:

1. One copy of the REFERENCE SHEET of questions for potential investigation (Table 6.1), with accompanying space to record a general Calculator Algorithm that covers the set of 12 questions. Such an algorithm is illustrated in Table 6.2 for the National Semiconductor "Mathematician" 4510 RPN calculator used by students in the accelerated seventh grade class.

2. For each question investigated by a student, another copy of the REFERENCE SHEET on which the student can record an algorithm particularized for the question being investigated. This is illustrated in Table 6.3 for question #10.

3. A copy of the RECORD SHEET (Table 6.4), also to be particularized for the question being investigated, as illustrated in Table 6.5 for question #10. For each data line on a RECORD SHEET the student is to indicate whether $d = c$ or $d \neq c$.

Variation can be expected in the degree to which teacher guidance is helpful or necessary for students to reach the conclusions summarized in Table 6.6, with particular attention to operations and conditions for which $\boxed{\nabla}$ is distributive over $\boxed{\circ}$.

User Instructions and the accompanying program are included in Appendix 6A for persons who may wish to use an EduCALC 29GD when working with students on this unit.

TABLE 6.1

REFERENCE SHEET

$(a \circ b)\boxed{\nabla} = c$ and $(a\boxed{\nabla}) \circ (b\boxed{\nabla}) = d$

Under what condition(s) for a and b is it true that $d = c$ when:

# 1. $(a + b)\boxed{x^2} = c$ and $(a\boxed{x^2}) + (b\boxed{x^2}) = d$?
# 2. $(a - b)\boxed{x^2} = c$ and $(a\boxed{x^2}) - (b\boxed{x^2}) = d$?
# 3. $(a \times b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \times (b\boxed{x^2}) = d$?
# 4. $(a \div b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \div (b\boxed{x^2}) = d$?

# 5. $(a + b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) + (b\boxed{\sqrt{\ }}) = d$?
# 6. $(a - b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) - (b\boxed{\sqrt{\ }}) = d$?
# 7. $(a \times b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) \times (b\boxed{\sqrt{\ }}) = d$?
# 8. $(a \div b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) \div (b\boxed{\sqrt{\ }}) = d$?

# 9. $(a + b)\boxed{1/x} = c$ and $(a\boxed{1/x}) + (b\boxed{1/x}) = d$?
#10. $(a - b)\boxed{1/x} = c$ and $(a\boxed{1/x}) - (b\boxed{1/x}) = d$?
#11. $(a \times b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \times (b\boxed{1/x}) = d$?
#12. $(a \div b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \div (b\boxed{1/x}) = d$?

Calculator Algorithm

Step	Input	Key(s)	Output
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	
		☐ ☐	

TABLE 6.2

REFERENCE SHEET

$(a \circ b)\boxed{\nabla} = c$ and $(a\boxed{\nabla}) \circ (b\boxed{\nabla}) = d$

Under what condition(s) for a and b is it true that $d = c$ when:

# 1. $(a + b)\boxed{x^2} = c$ and $(a\boxed{x^2}) + (b\boxed{x^2}) = d$?

# 2. $(a - b)\boxed{x^2} = c$ and $(a\boxed{x^2}) - (b\boxed{x^2}) = d$?

# 3. $(a \times b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \times (b\boxed{x^2}) = d$?

# 4. $(a \div b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \div (b\boxed{x^2}) = d$?

# 5. $(a + b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) + (b\boxed{\sqrt{\ }}) = d$?

# 6. $(a - b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) - (b\boxed{\sqrt{\ }}) = d$?

# 7. $(a \times b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) \times (b\boxed{\sqrt{\ }}) = d$?

# 8. $(a \div b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) \div (b\boxed{\sqrt{\ }}) = d$?

# 9. $(a + b)\boxed{1/x} = c$ and $(a\boxed{1/x}) + (b\boxed{1/x}) = d$?

#10. $(a - b)\boxed{1/x} = c$ and $(a\boxed{1/x}) - (b\boxed{1/x}) = d$?

#11. $(a \times b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \times (b\boxed{1/x}) = d$?

#12. $(a \div b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \div (b\boxed{1/x}) = d$?

General
Calculator Algorithm

Step	Input	Key(s)		Output
1	a	ENT	ENT	a
2	b	MS		b
3		$\circ$		$a \circ b$
4		∇		c
5		C↓	∇	$a\boxed{\nabla}$
6		MR	∇	$b\boxed{\nabla}$
7		$\circ$		d

TABLE 6.3

REFERENCE SHEET

$(a \circ b)\boxed{\nabla} = c$ and $(a\boxed{\nabla}) \circ (b\boxed{\nabla}) = d$

Under what condition(s) for a and b is it true that $d = c$ when:

1. $(a + b)\boxed{x^2} = c$ and $(a\boxed{x^2}) + (b\boxed{x^2}) = d$?
2. $(a - b)\boxed{x^2} = c$ and $(a\boxed{x^2}) - (b\boxed{x^2}) = d$?
3. $(a \times b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \times (b\boxed{x^2}) = d$?
4. $(a \div b)\boxed{x^2} = c$ and $(a\boxed{x^2}) \div (b\boxed{x^2}) = d$?

5. $(a + b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) + (b\boxed{\sqrt{\ }}) = d$?
6. $(a - b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) - (b\boxed{\sqrt{\ }}) = d$?
7. $(a \times b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) \times (b\boxed{\sqrt{\ }}) = d$?
8. $(a \div b)\boxed{\sqrt{\ }} = c$ and $(a\boxed{\sqrt{\ }}) \div (b\boxed{\sqrt{\ }}) = d$?

9. $(a + b)\boxed{1/x} = c$ and $(a\boxed{1/x}) + (b\boxed{1/x}) = d$?
#10. $(a - b)\boxed{1/x} = c$ and $(a\boxed{1/x}) - (b\boxed{1/x}) = d$?
#11. $(a \times b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \times (b\boxed{1/x}) = d$?
#12. $(a \div b)\boxed{1/x} = c$ and $(a\boxed{1/x}) \div (b\boxed{1/x}) = d$?

Calculator Algorithm

for Question # 10

Step	Input	Key(s)		Output
1	a	ENT	ENT	a
2	b	MS		b
3		-		$a - b$
4		$1/x$		c
5		C↓	$1/x$	$a\boxed{1/x}$
6		MR	$1/x$	$b\boxed{1/x}$
7		-		d

TABLE 6.4
RECORD SHEET

$(a \circ b)\boxed{\nabla} = c$ and $(a\boxed{\nabla}) \circ (b\boxed{\nabla}) = d$

#_____: $(a \quad b)\boxed{} = c$ and $(a\boxed{}) \quad (b\boxed{}) = d$

a	b	$(a \quad b)$	c	$a\boxed{}$	$b\boxed{}$	d	d ? c
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$

TABLE 6.5

RECORD SHEET

$(a \circ b)\boxed{\nabla} = c$ and $(a\boxed{\nabla}) \circ (b\boxed{\nabla}) = d$

10 : $(a - b)\boxed{1/x} = c$ and $(a\boxed{1/x}) - (b\boxed{1/x}) = d$

a	b	$(a - b)$	c	$a\boxed{1/x}$	$b\boxed{1/x}$	d	$d \; ? \; c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$
							$d \quad c$

TABLE 6.6

c d	c d	c d
$((a + b)\boxed{x^2}) = (a\boxed{x^2} + b\boxed{x^2})$	$((a + b)\boxed{\sqrt{\ }}) = (a\boxed{\sqrt{\ }} + b\boxed{\sqrt{\ }})$	$((a + b)\boxed{1/x}) = (a\boxed{1/x} + b\boxed{1/x})$
[1] All a, $b = 0$ [2] All b, $a = 0$	[1] All $a \geq 0$, $b = 0$ [2] All $b \geq 0$, $a = 0$	
$((a - b)\boxed{x^2}) = (a\boxed{x^2} - b\boxed{x^2})$	$((a - b)\boxed{\sqrt{\ }}) = (a\boxed{\sqrt{\ }} - b\boxed{\sqrt{\ }})$	$((a - b)\boxed{1/x}) = (a\boxed{1/x} - b\boxed{1/x})$
[1] All a, $b = 0$ [2] All $a = b$	[1] All $a \geq 0$, $b = 0$ [2] All $(a \geq 0) = b$	
$((a \times b)\boxed{x^2}) = (a\boxed{x^2} \times b\boxed{x^2})$	$((a \times b)\boxed{\sqrt{\ }}) = (a\boxed{\sqrt{\ }} \times b\boxed{\sqrt{\ }})$	$((a \times b)\boxed{1/x}) = (a\boxed{1/x} \times b\boxed{1/x})$
[1] All a, b	[1] All $a \geq 0$, $b \geq 0$	[1] All $a \neq 0$, $b \neq 0$
$((a \div b)\boxed{x^2}) = (a\boxed{x^2} \div b\boxed{x^2})$	$((a \div b)\boxed{\sqrt{\ }}) = (a\boxed{\sqrt{\ }} \div b\boxed{\sqrt{\ }})$	$((a \div b)\boxed{1/x}) = (a\boxed{1/x} \div b\boxed{1/x})$
[1] All a, $b \neq 0$	[1] All $a \geq 0$, $b > 0$	[1] All $a \neq 0$, $b \neq 0$

For each of the preceding conditions, a and b are rational numbers.

APPENDIX 6A

EduCALC 29GD User Instructions

STEP	INSTRUCTIONS	INPUT DATA/UNITS	Key(s)		OUTPUT DATA/UNITS
1	Set display: p places:		f FIX	p	
	(May be changed any time)				
2	Set program pointer:		g	RTN	
3	Do ONE of these:				
	If unary op. is x^2:	1	STO	9	
	If unary op. is $\sqrt{x}$:	2	STO	9	
	If unary op. is $1/x$:	3	STO	9	
4	Do ONE of these:				
	If binary op. is + :	5	STO	0	
	If binary op. is - :	6	STO	0	
	If binary op. is × :	7	STO	0	
	If binary op. is ÷ :	8	STO	0	
5	Iterate steps 6 thru 10				
	for each a and b.				
6	Input a, b; calc. $a \circ b$:	a	ENTER↑		
		b	R/S		$a \circ b$
7	Calculate c:		R/S		c
8	Calculate $a\boxed{\nabla}$:		R/S		$a\boxed{\nabla}$
9	Calculate $b\boxed{\nabla}$:		R/S		$b\boxed{\nabla}$
10	Calculate d:		R/S		d

NOTE: Step 3 or/and Step 4 may be changed at the end of any iteration of Steps 6 thru 10 without disrupting program execution. If the display gives an Error message at any time, steps 2, 3, and 4 must be re-executed before continuing.

EduCALC GD29 Program

Key(s)	Line	Code
g LBL 0	01	15 13 00
STO 2	02	23 02
$x \rightleftarrows y$	03	21
STO 1	04	23 01
$x \rightleftarrows y$	05	21
GSB i	06	12 22
R/S	07	74
GSB 9	08	12 09
R/S	09	74
RCL 1	10	24 01
GSB i	11	12 22
R/S	12	74
RCL 2	13	24 02
GSB i	14	12 22
R/S	15	74
g LBL 9	16	15 13 09
RCL 9	17	24 09
RCL 0	18	24 00
STO 9	19	23 09
R↓	20	22
STO 0	21	23 00
R↓	22	22
GSB i	23	12 22
g RTN	24	15 12
GTO 0	25	13 00
g LBL 1	26	15 13 01
g x^2	27	15 63
g RTN	28	15 12
g LBL 2	29	15 13 02
f $\sqrt{x}$	30	14 63
g RTN	31	15 12
g LBL 3	32	15 13 03
g $1/x$	33	15 74
g RTN	34	15 12
g LBL 5	35	15 13 05
+	36	51
g RTN	37	15 12
g LBL 6	38	15 13 06
-	39	41
g RTN	40	15 12
g LBL 7	41	15 13 07
×	42	61
g RTN	43	15 12
g LBL 8	44	15 13 08
÷	45	71
g RTN	46	15 12

VII

CALCULATORS AND UNARY OPERATIONS

by

J. Fred Weaver

An abridged version of this chapter has been accepted for 1979 publication in School Science and Mathematics.

Several years ago, Hess (1974) indicated that school mathematics programs and texts give much emphasis to binary operations (diadic functions) but relatively little attention to unary operations (monadic functions), which may prompt students to conclude that binary operations "are the only important operations in mathematics [p. 281]." This condition may have changed somewhat within the past few years, but not markedly so.

Unary operations on numbers are encountered in a variety of calculations. More important mathematically, however, is the fact that unary operations have properties and distinguishing characteristics just as do binary operations.

Systematic consideration of selected unary operations can be facilitated by using electronic calculators. Included in this chapter are suggestions and illustrations of some of the ways in which this can be accomplished. Emphasis is placed upon unary operations that are particularly suitable for exploration and investigation at a pre-algebra level, using calculator algorithms as the basis for generating examples or nonexamples to develop or test conjectures.

Domain of Definition

Hess (1974) indicated, "Unary operations and binary operations are particular cases of n-ary operations [p. 281]." A student may be well along in the study of mathematics, however, before comprehending the preceding statement in these terms (Lay, 1966):

> Let us now consider a rule of correspondence that assigns to ordered subsets of n elements of S a unique element of T. This will be called an n-ary operation from the set S into the set T. If this correspondence applies to every possible ordered subset of n elements in S, the operation

> is said to be complete, or performable on S. If the operation is complete and T = S, then the operation is closed on S. Since both the domain of definition and the range of images are now in S, we say we have an n-ary operation in S. According to whether n = 1, 2, 3, . . ., n the operation is said to be unary, binary, ternary, . . ., n-ary [p. 198].

At times, especially at a pre-algebra level, it is important and necessary to distinguish between what I shall call the canonical and the expedient domains of a unary operation. The former refers to the domain over which a particular operation customarily is defined within the algebra of the real number system; the latter, to a domain that is a proper subset of the canonical domain.

For example, the squaring operation, $a \rightarrow a^2$, has as its canonical domain the set of real numbers. However, at a particular instructional level this unary operation may have for its expedient domain the set of non-negative integers, or the set of rational numbers, etc., simply because at a given instructional level students' knowledge of number does not go beyond some distinguished proper subset of the reals. There are some unary functions, of course, for which the canonical domain itself is some proper subset of the reals; e.g., $a \rightarrow \log a$.

As teachers use any of these instructional suggestions, they will have to decide which domains--canonical or expedient--are appropriate within their own classroom situations.

Notes on Notation

Unary operations or monadic functions are customarily symbolized in a variety of ways, with different notations used at times for the same operation or function. For example:

$\log x$

$\sqrt{x}$

$|x|$

$[x]$

$x!$

x^2

x^{-1} or $\frac{1}{x}$ or $1/x$

e^x

As Gilman & Rose (1976) have indicated,

> Whatever other merits this mishmash has, consistency certainly isn't one of them, for the symbol which is the functional indicator may appear on the left, the right, both sides, on top, or be in a special position, or be represented by an alphabetical label [p. 57].

Such inconsistency is intolerable in a computer language. One way to resolve this situation is to develop a special set of symbols so that unary or monadic functions may be indicated consistently in a form such as "∇a" which is termed prefix or pre-operator or left-operator notation. The unary operator (∇) is written immediately before the argument or operand (a). This is the resolution adopted by APL (A Programming Language).

Another way to resolve the situation is to use a notation of the form "$a \nabla$" which is termed postfix or post-operator or right-operator notation in which the unary operator (∇) is written immediately after the argument or operand (a). With minor modification this is the notational form used here and is judged to be the preferable one for a calculator language, regardless of whether the calculator procedures for data entry and processing are based upon algebraic logic (with or without an operational hierarchy) or upon RPN (Reverse Polish Notation). This does not resolve all of the notational issues identified in the Report of the Conference on Needed Research and Development on Hand-held

Calculators in School Mathematics [n.d., p. 32], but it does provide a notational consistency across calculators when working with unary or monadic operations or functions.

These are symbolized in the general form "$a\,\boxed{\nabla}$ " where a is the argument or operand and $\boxed{\nabla}$ is a facsimile of a particular calculator key, with ∇ replaced by the symbol on the face of that key. This is consistent with the fact that calculator-computation of the value of a particular monadic function applied to a given argument is effected by first entering and/or displaying the operand (a) and then activating the appropriate operator key ($\boxed{\nabla}$).

Emphasis is given to four monadic functions, each with the indicated notation for calculator execution of that function:

1. The *squaring* function (i.e., $a \to a^2$) .... $a\,\boxed{x^2}$
2. The *principal square root* function (i.e., $a \to \sqrt{a}$) .... $a\,\boxed{\sqrt{x}}$
3. The *additive inverse* function (i.e., $a \to {}^{-}a$) .... $a\,\boxed{\text{CHS}}$
4. The *reciprocal* or *multiplicative inverse* function (i.e., $a \to \frac{1}{a}$ or $a \to a^{-1}$) .... $a\,\boxed{1/x}$

Although some calculators have a $\boxed{+/-}$ key rather than a $\boxed{\text{CHS}}$ key for function 3, I personally prefer the latter notation for the intended conceptualization. It also is recognized that in some of the early electronic calculators such as Hewlett-Packard's model HP-35, the $\boxed{\text{CHS}}$ key or its equivalent could be executed as either a pre- or post-operator; or in the case of some calculators, as a pre- rather than post-operator. To the best of my knowledge, current models of calculators use exclusively a post-operator execution position for this additive inverse function.

Less emphasis is given to the following monadic functions:

5. **The *common logarithm* function (i.e., $a \to \log a$) ...... $a\boxed{\log}$**

6. The *common antilog* function (i.e., $a \to 10^a$) .... $a\boxed{10^x}$

Function 6 assumes a calculator with a hard-wired $\boxed{10^x}$ key. If execution had to be effected using a $\boxed{y^x}$ key in conjunction with keyboard entry of both a and 10, some people would contend that now we are dealing with a diadic or binary operation rather than a monadic operation.

Work with the furst four of the preceding unary operators is appropriate for, and has been carried out with, pre-algebra students at the middle or junior high school level. Wirk with the latter two unary operators is more suitable in connection with algebra rather than pre-algebra courses.

Notation-related Matters

Two notation-related matters are identified and discussed.

1. The calculator notation for unary operations mirrors the keystroke sequences used in evaluating exemplars of such functions. In the case of some calculators, however, a shift or prefix key must be used prior to an operator key in order to select the desired operator associated with a two- or three-function key. However, such a shift or prefix key is not incorporated in the notation used here.

2. The problems associated with interpreting an expression such as "-(-5)," discussed by Musser (1972) and termed by Van Engen (1972) as "just a fact of life for a mathematics teacher [p. 299]," are not eliminated when using a calculator; in fact, they may be magnified by use of the $\boxed{\text{CHS}}$ key. For instance, in order to evaluate "-5 $\boxed{\text{CHS}}$,"

the following keystroke sequence is used: $5\ \boxed{\text{CHS}}\boxed{\text{CHS}}$, in which the two $\boxed{\text{CHS}}$ keystrokes accomplish conceptually different things. It is important to distinguish between the $\boxed{\text{CHS}}$ operator applied to operand a, and an implicit $\boxed{\text{CHS}}$ calculator keystroke executed to input an operand $a < 0$, to which $\boxed{\text{CHS}}$ or some other operator is then applied.

Instructional Suggestions

The material that follows is intended simply to suggest avenues for instructional exploration and investigation. It is assumed that teachers will exercise considerable latitude in the explicit ways in which the suggestions may be incorporated into ongoing instructional programs.

One thing in particular should be kept in mind: Since calculators often generate and display rational approximations to real numbers rather than real numbers per se, this condition must be handled accordingly when attempting to ascertain the validity of conjectured properties from specific numerical exemplars. The consequences of rounding or truncating must be recognized and taken into account.

Graphs

The construction of a portion of a graph of each of the following functions (assuming a to be a real number) can be helpful in getting an initial understanding for the unary operators involved--their similarities and their differences--and also in dealing with some subsequent considerations:

1. $a \rightarrow a\,\boxed{x^2}$
2. $a \rightarrow a\,\boxed{\sqrt{x}}$
3. $a \rightarrow a\,\boxed{\text{CHS}}$
4. $a \rightarrow a\,\boxed{1/x}$

In Figures 7.1 and 7.2, the partial graphs for the latter two preceding functions are markedly different from each other. How do the graphs of the first two of the preceding functions compare with each other and with Figures 7.1 and 7.2?

Relations

Some unary operations maintain the sense of certain binary relations. For all a,b that make the antecedent true, is the consequent also true? If not, under what condition(s) would it be true?

1. $a = b \Rightarrow a\boxed{x^2} = b\boxed{x^2}$
2. $a > b \Rightarrow a\boxed{x^2} > b\boxed{x^2}$
3. $a < b \Rightarrow a\boxed{x^2} < b\boxed{x^2}$
4. $a = b \Rightarrow a\boxed{\sqrt{x}} = b\boxed{\sqrt{x}}$
5. $a > b \Rightarrow a\boxed{\sqrt{x}} > b\boxed{\sqrt{x}}$
6. $a < b \Rightarrow a\boxed{\sqrt{x}} < b\boxed{\sqrt{x}}$
7. $a = b \Rightarrow a\boxed{\text{CHS}} = b\boxed{\text{CHS}}$
8. $a > b \Rightarrow a\boxed{\text{CHS}} > b\boxed{\text{CHS}}$
9. $a < b \Rightarrow a\boxed{\text{CHS}} < b\boxed{\text{CHS}}$
10. $a = b \Rightarrow a\boxed{1/x} = b\boxed{1/x}$
11. $a > b \Rightarrow a\boxed{1/x} > b\boxed{1/x}$
12. $a < b \Rightarrow a\boxed{1/x} < b\boxed{1/x}$

Inverses

Some unary operations are inverses of each other or of themselves. Is the sentence true for all a? If not, under what condition(s) is the sentence true?

1. $a\boxed{x^2}\,\boxed{\sqrt{x}} = a$
2. $a\boxed{\sqrt{x}}\,\boxed{x^2} = a$
3. $a\boxed{\text{CHS}}\,\boxed{\text{CHS}} = a$
4. $a\boxed{1/x}\,\boxed{1/x} = a$

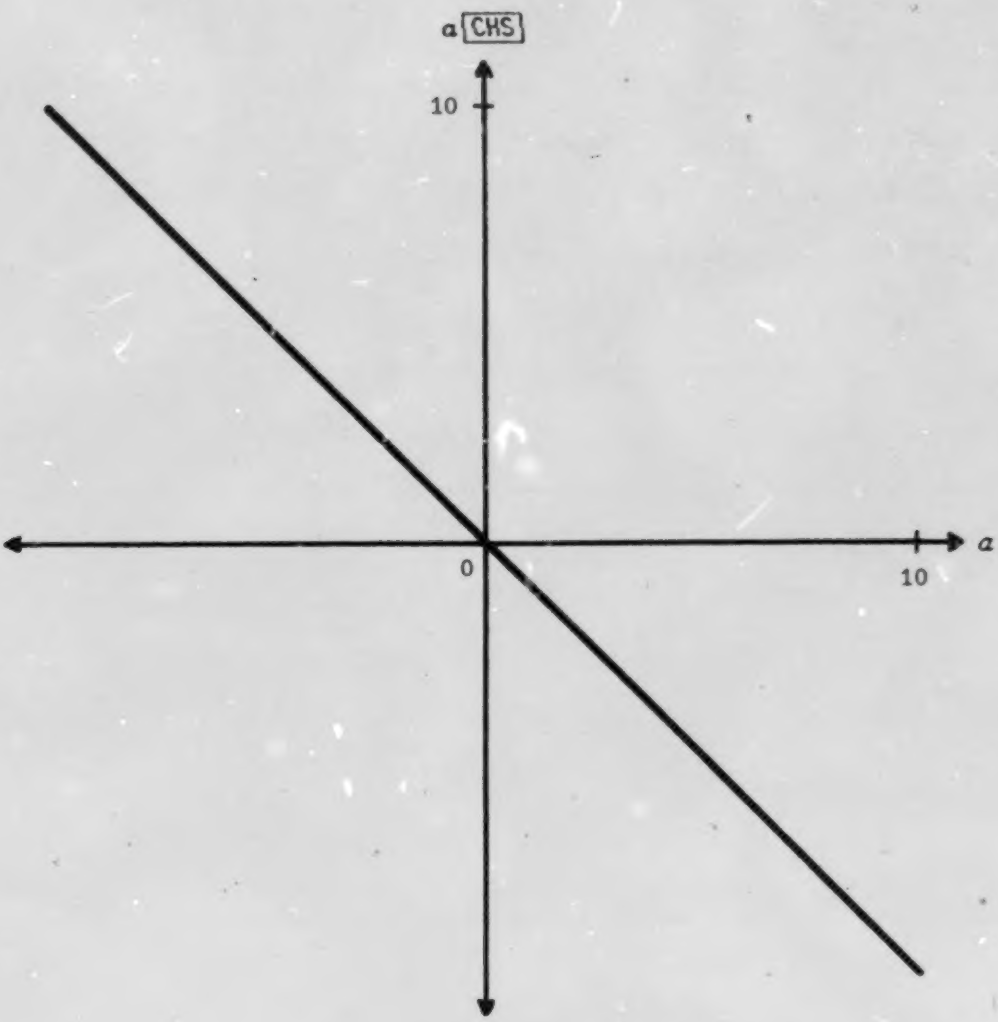

Figure 7.1. Partial graph of $a \rightarrow a\boxed{\text{CHS}}$

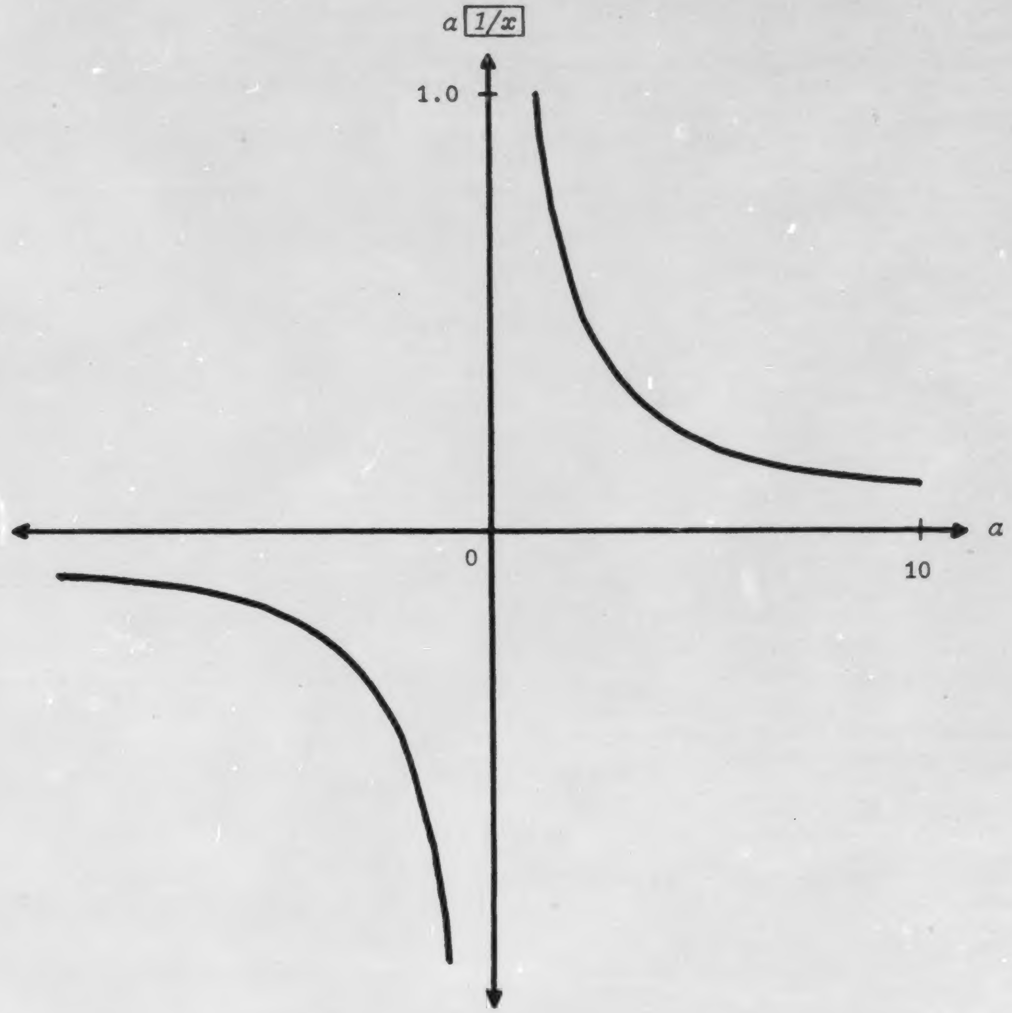

Figure 7.2. Partial graph of $a \rightarrow a\,\boxed{1/x}$

Commutativity

Is the sentence true for all a? If not, under what condition(s) is it true?

1. $a\ \boxed{x^2}\ \boxed{\text{CHS}} = a\ \boxed{\text{CHS}}\ \boxed{x^2}$
2. $a\ \boxed{x^2}\ \boxed{1/x} = a\ \boxed{1/x}\ \boxed{x^2}$
3. $a\ \boxed{\sqrt{x}}\ \boxed{\text{CHS}} = a\ \boxed{\text{CHS}}\ \boxed{\sqrt{x}}$
4. $a\ \boxed{\sqrt{x}}\ \boxed{1/x} = a\ \boxed{1/x}\ \boxed{\sqrt{x}}$
5. $a\ \boxed{\text{CHS}}\ \boxed{1/x} = a\ \boxed{1/x}\ \boxed{\text{CHS}}$

Distributivity

Is the sentence true for all a,b? If not, under what condition(s) is it true?

1. $(a + b)\ \boxed{x^2} = a\ \boxed{x^2} + b\ \boxed{x^2}$
2. $(a - b)\ \boxed{x^2} = a\ \boxed{x^2} - b\ \boxed{x^2}$
3. $(a \times b)\ \boxed{x^2} = a\ \boxed{x^2} \times b\ \boxed{x^2}$
4. $(a \div b)\ \boxed{x^2} = a\ \boxed{x^2} \div b\ \boxed{x^2}$
5. $(a + b)\ \boxed{\sqrt{x}} = a\ \boxed{\sqrt{x}} + b\ \boxed{\sqrt{x}}$
6. $(a - b)\ \boxed{\sqrt{x}} = a\ \boxed{\sqrt{x}} - b\ \boxed{\sqrt{x}}$
7. $(a \times b)\ \boxed{\sqrt{x}} = a\ \boxed{\sqrt{x}} \times b\ \boxed{\sqrt{x}}$
8. $(a \div b)\ \boxed{\sqrt{x}} = a\ \boxed{\sqrt{x}} \div b\ \boxed{\sqrt{x}}$
9. $(a + b)\ \boxed{\text{CHS}} = a\ \boxed{\text{CHS}} + b\ \boxed{\text{CHS}}$
10. $(a - b)\ \boxed{\text{CHS}} = a\ \boxed{\text{CHS}} - b\ \boxed{\text{CHS}}$
11. $(a \times b)\ \boxed{\text{CHS}} = a\ \boxed{\text{CHS}} \times b\ \boxed{\text{CHS}}$
12. $(a \div b)\ \boxed{\text{CHS}} = a\ \boxed{\text{CHS}} \div b\ \boxed{\text{CHS}}$
13. $(a + b)\ \boxed{1/x} = a\ \boxed{1/x} + b\ \boxed{1/x}$
14. $(a - b)\ \boxed{1/x} = a\ \boxed{1/x} - b\ \boxed{1/x}$
15. $(a \times b)\ \boxed{1/x} = a\ \boxed{1/x} \times b\ \boxed{1/x}$
16. $(a \div b)\ \boxed{1/x} = a\ \boxed{1/x} \div b\ \boxed{1/x}$

Exemplars for each of the preceding 16 sentence types may be generated from a *calculator algorithm* as suggested by Table 7.1 or Table 7.2 (depending upon the kind of calculator used), where $\boxed{\circ}$ and $\boxed{\nabla}$ may be replaced by some particular binary and unary operator respectively. In each such instance, is $c = d$ for all a,b ? (Or, the algorithms could be extended very readily to answer the equivalent question: Is $c - d = 0$?

TABLE 7.1

An Algebraic-logic Calculator Algorithm for Distributivity Sentences

Left-hand member				Right-hand member			
INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
a	$\boxed{\circ}$	$\boxed{}$		a	$\boxed{\nabla}$	$\boxed{\circ}$	
b	$\boxed{=}$	$\boxed{\nabla}$	c	b	$\boxed{\nabla}$	$\boxed{=}$	d

TABLE 7.2

An RPN Calculator Algorithm for Distributivity Sentences

Left-hand member				Right-hand member			
INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS	INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
a	$\boxed{\text{ENTER}\uparrow}$	$\boxed{}$		a	$\boxed{\nabla}$	$\boxed{}$	
b	$\boxed{\circ}$	$\boxed{\nabla}$	c	b	$\boxed{\nabla}$	$\boxed{\circ}$	d

Iterations

An iteration of the general form

$$a\boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}\ \cdots$$

may be studied by generating, term by term, the following sequence:

$$n_1 = a_i\boxed{\nabla}$$

$$n_2 = a_i\boxed{\nabla}\ \boxed{\nabla}$$

$$n_3 = a_i\boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}$$

$$n_4 = a_i\boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}$$

$$n_5 = a_i\boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}\ \boxed{\nabla}$$

etc.

for some particular $\boxed{\nabla}$ and a_i, and then across sequences generated from the same $\boxed{\nabla}$ applied to various a_i's.

In what way(s), if at all, do iterations 1 and 2 below "behave" more or less similarly for all a ?

1. $a\boxed{\text{CHS}}\ \boxed{\text{CHS}}\ \boxed{\text{CHS}}\ \boxed{\text{CHS}}\ \boxed{\text{CHS}}\ \cdots$
2. $a\boxed{1/x}\ \boxed{1/x}\ \boxed{1/x}\ \boxed{1/x}\ \boxed{1/x}\ \cdots$

Can the following iteration 3 be characterized in the same way for all a ?

3. $a\boxed{x^2}\ \boxed{x^2}\ \boxed{x^2}\ \boxed{x^2}\ \boxed{x^2}\ \cdots$

In what way(s) is iteration 4 below for all $0 < a < 1$ similar to that for all $a > 1$?

4. $a\boxed{\sqrt{x}}\ \boxed{\sqrt{x}}\ \boxed{\sqrt{x}}\ \boxed{\sqrt{x}}\ \boxed{\sqrt{x}}\ \cdots$

Although a bit more complex in its structure, a much more mathematically interesting iteration involving the $\boxed{\sqrt{x}}$ operator may be generated progressively in the following manner:

5. $n_1 = a\boxed{\sqrt{x}}$

$n_2 = (a\boxed{\sqrt{x}} + a)\boxed{\sqrt{x}}$

$n_3 = ((a\boxed{\sqrt{x}} + a)\boxed{\sqrt{x}} + a)\boxed{\sqrt{x}}$

$n_4 = (((a\boxed{\sqrt{x}} + a)\boxed{\sqrt{x}} + a)\boxed{\sqrt{x}} + a)\boxed{\sqrt{x}}$

etc.

That is to say, we are dealing with:

$$n = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}}$$

For a given a, convergence to some number n is relatively rapid. It might be anticipated that n is necessarily irrational. But this is not necessarily the case. With a suitable choice for a, n can be made to approach an integer; not just some integer, but any desired integer greater than 1. It can be shown that $a = n(n - 1)$ [Ogilvy & Anderson, 1966, p. 62]. Pick the integer $n > 1$; then calculate the a to be used.

For example, convergence to $n = 3$ can be effected by using $a = 3(3 - 1) = 6$. This is illustrated in Table 7.3 which translates the iterative procedure into keystrokes for representative calculators.[1]

---

[1] In the keystroke sequence, it should be noted that (1) some algebraic-logic calculators, unlike the TI-1270 and TI-30, have a four-key memory such as: $\boxed{\text{CM}}$, $\boxed{\text{MR}}$, $\boxed{\text{M+}}$ and $\boxed{\text{M-}}$. For such calculators, $\boxed{\text{CM}}$ $\boxed{\text{M+}}$ would replace $\boxed{\text{STO}}$, and $\boxed{\text{MR}}$ would be the same as $\boxed{\text{RCL}}$ in the specified keystroke sequence; (2) some RPN calculators such as National Semiconductor's "Mathematician" and "Scientist" models do not permit stack storage as does the HP-21, due to a difference in the behavior of the top stack register when the stack drops. This can be accommodated, however, in such non-HP calculators by a slightly more involved routine of data manipulations.

TABLE 7.3

Exemplar for Iterations: 5

TI-1270 or TI-30 keystrokes	HP-21 keystrokes	Truncated display	Iteration approximated
6 $\boxed{\text{STO}}$	6 $\boxed{\text{ENTER}\uparrow}$ $\boxed{\text{ENTER}\uparrow}$ $\boxed{\text{ENTER}\uparrow}$	6.00000	
$\boxed{\sqrt{x}}$	■ $\boxed{\sqrt{x}}$	2.44948	$n_1 = \sqrt{6}$
$\boxed{+}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{+}$ ■ $\boxed{\sqrt{x}}$	2.90680	$n_2 = \sqrt{6 + \sqrt{6}}$
$\boxed{+}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{+}$ ■ $\boxed{\sqrt{x}}$	2.98442	$n_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$
$\boxed{+}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{+}$ ■ $\boxed{\sqrt{x}}$	2.99740	$n_4 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}$
$\boxed{+}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{+}$ ■ $\boxed{\sqrt{x}}$	2.99956	$n_5 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}}$
$\boxed{+}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{+}$ ■ $\boxed{\sqrt{x}}$	2.99992	$n_6 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}}}$
$\boxed{+}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{+}$ ■ $\boxed{\sqrt{x}}$	2.99998	$n_7 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}}}}$
$\boxed{+}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{+}$ ■ $\boxed{\sqrt{x}}$	2.99999	$n_8 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}}}}}$

Extensions

Further explorations can be made with unary operators. Also, other unary operators can be introduced and investigated. For example, the (common) log operator, and its inverse 10^x .

Given each of the following, under what condition(s) is it valid for all a, or for all a,b?

1. $a \boxed{\log} \boxed{10^x} = a$
2. $a \boxed{10^x} \boxed{\log} = a$
3. $(a + b) \boxed{\log} = a \boxed{\log} + b \boxed{\log}$
4. $(a - b) \boxed{\log} = a \boxed{\log} - b \boxed{\log}$
5. $(a \times b) \boxed{\log} = a \boxed{\log} \times b \boxed{\log}$
6. $(a \div b) \boxed{\log} = a \boxed{\log} \div b \boxed{\log}$

The following attempted iteration leads to consequences that often are not immediately evident to students at the outset:

7. $a \boxed{\log} \boxed{\log} \boxed{\log} \boxed{\log} \boxed{\log} \cdots$

Knowing a, can the maximum number of iterations be predicted?

Can valid iterations of the following nature be generated for $a < 0$?

8. $a \boxed{10^x} \boxed{10^x} \boxed{10^x} \boxed{10^x} \boxed{10^x} \cdots$

Conclusion

Unary operations are much more than things to be executed as occasion demands in the course of computation or calculation. Such operations have distinguishing properties as do binary operations. Using calculators in these ways can help students view unary operations in appropriate perspectives with attention to their significant characteristics as mathematical functions.

In connection with calculator algorithms, some form of postfix rather than prefix notation is deemed advisable for unary operators. Such notation reflects the point of operator execution in a keystroke sequence. Consider, for instance, evaluation of an expression of the form $\sqrt{a^2 + b^2}$.

Algebraic-logic calculator: $a\,\boxed{x^2}\,\boxed{+}\,b\,\boxed{x^2}\,\boxed{=}\,\boxed{\sqrt{x}}$.

RPN calculator: $a\,\boxed{x^2}\,b\,\boxed{x^2}\,\boxed{+}\,\boxed{\sqrt{x}}$.

Finally, it should be noted in the preceding algorithms that although an algebraic-logic calculator algorithm uses postfix notation for unary operators, a betweenfix notation is used for binary operators. In the case of RPN calculator algorithm, however, postfix notation is used for unary and binary operators alike.

This latter observation often is viewed as an advantage for RPN calculators and makes possible the simple flowchart shown in Figure 7.3 (HP-35 Operating Manual, 1973, p. 34), or the Figure 7.4 modification I used in connection with classroom instruction involving work with RPN calculators by accelerated seventh grade students in a pre-algebra course.[2]

---

[2] Two things should be noted about the flowcharts (Figures 7.3 and 7.4): (1) The number of levels or registers in a calculator's operational stack--usually 4, sometimes 3--places a practical limit upon the number of operands that can be held in storage prior to the execution of a unary or binary operation. The flowcharts, as drawn, do not take this limitation into account and imply a stack of infinitely many levels or registers. (2) The flowcharts are not intended to cover data storage in addressable registers, or other data manipulations--functions that may be needed when using a calculator with a 4- or 3-level operational stack, but not with an idealized infinitely-many-level stack.

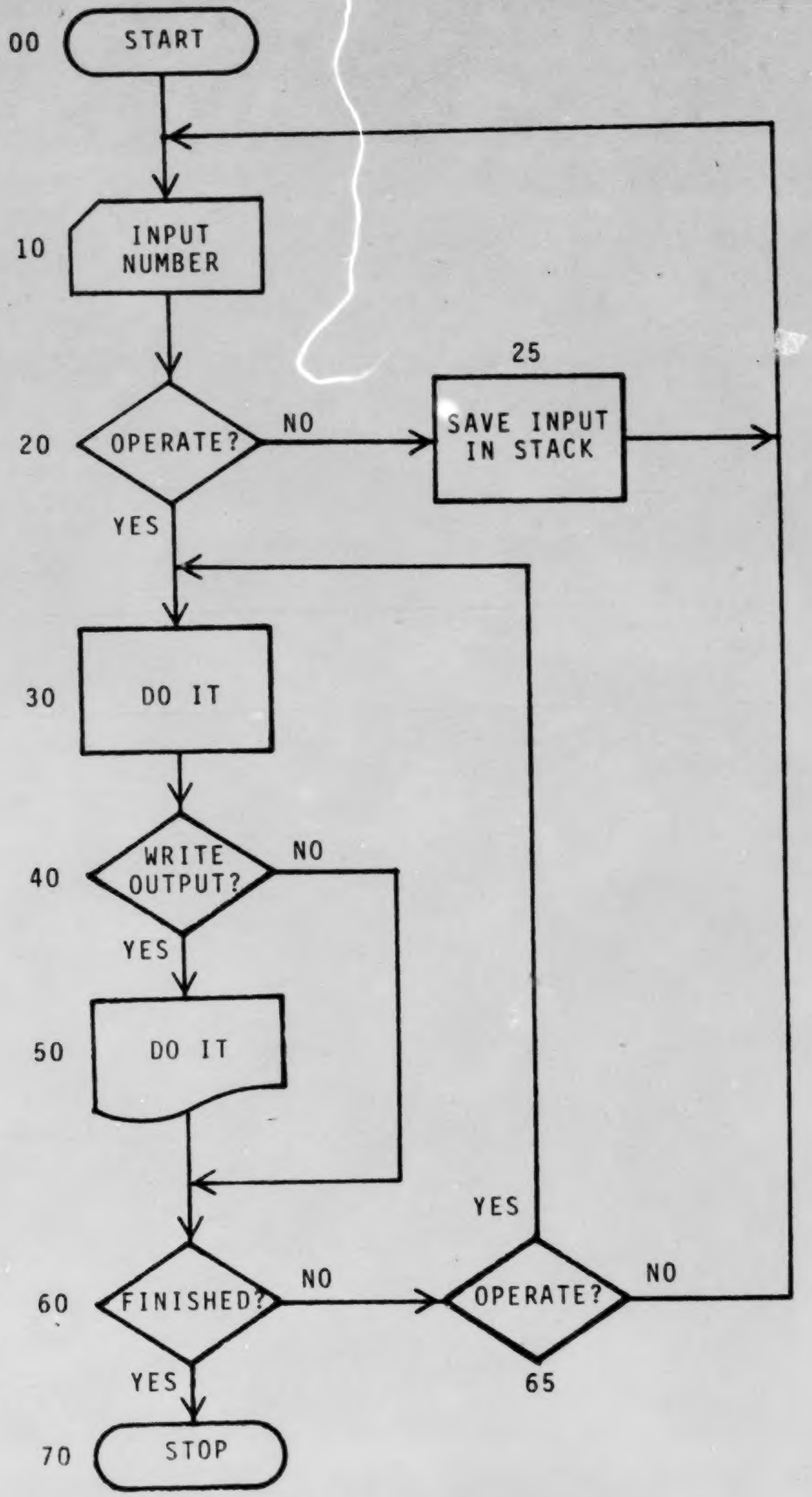
00
START
10
INPUT NUMBER
20
OPERATE?
NO
25
SAVE INPUT IN STACK
YES
30
DO IT
40
WRITE OUTPUT?
NO
YES
50
DO IT
60
FINISHED?
NO
YES
OPERATE?
65
YES
NO
70
STOP

Figure 7.3. A modification.

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VIII

STUDENT PERFORMANCE USING RPN-BASED CALCULATORS TO EVALUATE NUMERICAL EXPRESSIONS

by

J. Fred Weaver

Toward the end of the 1977-78 school year, the following problem set was administered to students in the accelerated pre-algebra class, all of whom had been working with the NS-4510 RPN-based calculators:

Use your calculator to work each problem. Write the answer on this sheet. If an answer has more than 3 decimal places displayed, round-off the answer to 3 decimal places.

For each problem whose letter is encircled, fill out a record sheet to show the sequence of keystrokes you used to work that problem.

A. $((53 \times 79) - 1864) \div 305 =$

B. $4869 - ((35 + 97) \times 46) =$

C. $54.3 + (8.9 \times (28.7 \div 16.4)) =$

D. $(49 \times 64) - (8056 \div 64) =$

E. $4701 \div (8120 - (396 + 875)) =$

F. $2830 + (520900 \div 760) - 3150 =$

G. $43.21 - (87.65 - 56.78) \times 1.098 =$

H. $75 + 23 \times 67 - 462 \div 75 =$

I. $86.4 - (52.8 \div 12.9 + 8.3 \times 0.64) =$

J. $-29 \times ((58 - (-47)) \div (-35) =$

The problems were constructed to involve a variety of calculator manipulations and to permit alternative processing approaches. On half the test forms, items A, C, E, G, and I were identified as the ones for which a record of keystrokes should be recorded; and on half the test forms, items B, D, F, H, and J were so identified. The two versions of the test were distributed randomly among the 39 students tested. The form for recording keystrokes was familiar to the students (see Figure 8.1).

Findings

Complete data were available for 34 of the 39 students--17 for each of the two randomly-distributed versions of the test.

Table 8.1 summarizes the frequency of correct responses for the 10-item problem set and for the 5-item keystroke record. No basis of comparison exists upon which to judge the responses.

Table 8.2 summarizes student performance on an item-by-item basis in terms of correct numerical answers (10-item problem set) and correct keystroke sequences (5-item set). In the latter case, the correct answer may have been any one of several acceptable sequences for a given item.

The relatively low performance level on item B was due to the fact that many students simply chained the data input and sequence of operations. This was a valid procedure for item A.

Errors on item E occurred mainly from students' dividing by 4701, a consequence of failing to exchange data in X and Y registers prior to dividing.

CALCULATOR RECORD SHEET for

STEP	KEY		OUTPUT/DISPLAY	NOTE
	INPUT/NUMBER	OTHER		
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				

Figure 8.1. Calculator record sheet.

Table 8.1

Frequency Distributions of Correct Responses

Number of correct responses	Frequency: 10-item problem set	Frequency: Five-item keystroke record
10	6	
9	4	
8	6	
7	3	
6	7	
5	5	6
4	2	9
3	1	9
2		8
1		1
0		1
Total	34	34
Mean	7.1	3.2

The relatively low performance levels on items G, H, and I was attributed in each case to students' failure to apply the mathematical convention regarding order of operations. The "My Dear Aunt Sally Rule" was a part of students' regular instructional program and also was considered in connection with their calculator work.

Conclusion

The problem set used for this test deliberately placed a premium upon students' ability to manipulate data for various calculations when using an RPN-based calculator with a 3-level operational stack. Except for the principal error observed in connection with item E, students' incorrect responses generally seemed attributable to their _mis_interpretation of the sense of mathematical expressions rather than to erroneous calculator manipulations.

The likelihood is great that had students used algebraic-logic calculators with an operational hierarchy and nested-parentheses capability, as in the Texas Instruments TI-30 or SR-40 with its "AOS" (Algebraic Operating System), relatively few incorrect answers would have been observed. But that would not indicate that students necessarily comprehended the mathematics with which they were dealing. For instructional purposes, it is more desirable for a calculator to force student interpretation of expressions being evaluated than to make such thinking unnecessary.

Table 8.2

Percent of Correct Responses

Item	10-item problem set	Five-item keystroke record
A. ((53 × 79) - 1864) ÷ 305 =	97.1%	88.2%
B. 4869 - ((35 + 97) × 46) =	44.1%	41.1%
C. 54.3 + (8.9 × (28.7 ÷ 16.4)) =	100.0%	100.0%
D. (49 × 64) - (8056 ÷ 64) =	88.2%	76.5%
E. 4701 ÷ (8120 - (396 + 875)) =	64.7%	52.9%
F. 2830 + (520900 ÷ 760) - 3150 =	88.2%	76.5%
G. 43.21 - (87.65 - 56.78) × 1.098 =	47.1%	35.3%
H. 75 + 23 × 67 - 462 ÷ 75 =	52.9%	52.9%
I. 86.4 - (52.8 ÷ 12.9 + 8.3 × 0.64) =	55.9%	47.1%
J. -29 × ((58 - (-47)) ÷ (-35) =	76.5%	76.5%
All items	71.5%	64.7%

XI

CALCULATOR-BASED INSTRUCTION PERTAINING TO SIMPLE LINEAR EQUATIONS AND THEIR GRAPHS

by

J. Fred Weaver

Toward the end of the second year, several instructional periods were devoted to introductory explorations pertaining to simple linear equations and their graphs. The work was in two parts.

Part I. Given a linear equation of the form $ax + by = c$, students used a guess-and-test procedure (calculator-facilitated as needed) to ascertain values of x and y which did and did not satisfy a given equation. Values were recorded and corresponding points plotted on a record sheet (see Figure 9.1). Students observed that those points whose coordinates satisfied a given equation all fell on a line, whereas other points whose coordinates did not satisfy the given equation did not fall on the preceding line.

Part II. The work of Part I was extended, using sheets such as Figure 9.2, to include pairs of linear equations: simultaneous systems. Attention was given to systems for which the graphs of the pair of equations (1) intersected in one point, (2) did not intersect (were parallel), and (3) intersected in all points (were the same line).

This work was greatly facilitated by the instructor's use of several programmable calculators, software for which is included as the bulk of this chapter.

The instructional sessions were wholly introductory and exploratory, with no attempt to have students master any part of the material.

Table 9.1.

User Instructions for Part I

User Instructions

1 LEARNING ABOUT EQUATIONS and THEIR GRAPHS 2
Part I: $ax + by = c$

(Read side 1 and side 2 of program card)

		HP-19C/29C			HP-67/97			
STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
1	Input/store constants.				1	f	d	1
		a	STO	1	a	R/S		a
		b	STO	2	b	R/S		b
		c	STO	3	c	R/S		c
2	Execute/iterate 3, 4, 5 in any order.							
3	Does a selected pair x,y "fit" the equation?	x	ENT↑		x	ENT↑		
	Yes, if c' = c.	y	GSB	1	y	A		c'
4	For a selected x, what is y?	x	GSB	2	x	B		y
*								
5	For a selected y, what is x?	y	GSB	3	y	C		x
*								
6	For a new equation, return to 1.							
*	To verify these values, first:		GSB	0		D		x **
								y
	Then:		R/S			R/S		c
	** Displayed during a 1-second pause.							

Table 9.2

User Instructions for Part II

User Instructions

LEARNING ABOUT EQUATIONS and THEIR GRAPHS

Part II: $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$

(Read side 1 and side 2 of program card)

STEP	INSTRUCTIONS	HP-19C/29C INPUT DATA/UNITS	HP-19C/29C KEYS		HP-67/97 INPUT DATA/UNITS	HP-67/97 KEYS		OUTPUT DATA/UNITS
1	Input/store constants				1	f	d	1
		a_1	STO	1	a_1	R/S		a_1
		b_1	STO	2	b_1	R/S		b_1
		c_1	STO	3	c_1	R/S		c_1
		a_2	STO	4	a_2	R/S		a_2
		b_2	STO	5	b_2	R/S		b_2
		c_2	STO	6	c_2	R/S		c_2
	OR, if constants from the one equation have been stored previously:				4	f	d	4
		a_2	STO	4	a_2	R/S		a_2
		b_2	STO	5	b_2	R/S		b_2
		c_2	STO	6	c_2	R/S		c_2
2	Execute/iterate 3 thru 9 in any order.							
3.1	Does a selected x,y fit Equation 1?	x	ENT↑		x	ENT↑		
	Yes, if $c' = c_1$	y	GSB	1	y	A		c'
3.2	Does that x,y fit Equation 2? First do this:		GSB	0		D		x **
	Yes, if $c'' = c_2$							y
	Then do this:		GSB	4		f	a	c''
4.1	Does a selected x,y fit Equation 2?	x	ENT↑		x	ENT↑		
	Yes, if $c'' = c_2$	y	GSB	4	y	f	a	c''
4.2	Does that x,y fit Equation 1? First do this:		GSB	0		D		x **
	Yes, if $c' = c_1$							y
	Then do this:		R/S			R/S		c'
	(CONTINUED)							

Table 9.2 (continued)

User Instructions for Part II

User Instructions

LEARNING ABOUT EQUATIONS and THEIR GRAPHS
Part II (concluded)

		HP-19C/29C			HP-67/97			
STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS		INPUT DATA/UNITS	KEYS		OUTPUT DATA/UNITS
5.1	For a selected x, what y fits Equation 1?	x	GSB	2	x	B		y
5.2	Execute 3.2.							
6.1	For a selected y, what x fits Equation 1?	y	GSB	3	y	C		x
6.2	Execute 3.2.							
7.1	For a selected x, what y fits Equation 2?	x	GSB	5	x	f	b	y
7.2	Execute 4.2.							
8.1	For a selected y, what x fits equation 2?	y	GSB	6	y	f	c	x
8.2	Execute 4.2.							
	If an x,y that fits *both* Equation 1 and Equation 2 has not been found, do the following:							
9			GSB	9		E		x *
			R/S			R/S		y
*	If display shows ERROR, do this for 67/97: (1: 1 line; 2: 2 parallel lines)					CL x	f e	1 or 2
	OR do this for 19C/29C:		CL x	RCL 7				n_x
			RCL	8				n_y
	($n_x=n_y=0$: 1 line; otherwise, 2 parallel lines							
	To show that in the case of ERROR, d=0; do:		RCL	9		RCL	9	0
	** Displayed during 1-second pause.							

10. For a new pair of equations, return to 1.

Table 9.3

Part I and Part II Program for HP-19C/29C and 67/97

Program for HP-19C/29C

```
01 *LBL0
02 RCL7
03 PSE
04 RCL8
05 RTN
06 *LBL1
07 STO6
08 RCL2
09 ×
10 X⇄Y
11 STO7
12 RCL1
13 ×
14 +
15 RTN
16 *LBL2
17 STO7
18 RCL1
19 ×
20 RCL3
21 X⇄Y
22 -
23 RCL2
24 *LBL8
25 ÷
26 STO6
27 RTN
28 *LBL3
29 STO8
30 RCL2
31 ×
32 RCL3
33 X⇄Y
34 -
35 RCL1
36 *LBL7
37 ÷
38 STO7
39 RTN
40 *LBL4
41 STO8
42 RCL5
43 ×
44 X⇄Y
45 STO7
46 RCL4
47 ×
48 +
49 RTN
```

```
50 *LBL5
51 STO7
52 RCL4
53 ×
54 RCL6
55 X⇄Y
56 -
57 RCL5
58 STO8
59 *LBL6
60 STO8
61 RCL5
62 ×
63 RCL6
64 X⇄Y
65 -
66 RCL4
67 STO7
68 *LBL9
69 RCL1
70 RCL6
71 ×
72 RCL4
73 RCL3
74 ×
75 -
76 STO8
77 RCL5
78 RCL3
79 ×
80 RCL2
81 RCL6
82 ×
83 -
84 STO7
85 RCL1
86 RCL5
87 ×
88 RCL4
89 RCL2
90 ×
91 -
92 STO9
93 ÷
94 R/S
95 RCL6
96 RCL9
97 ÷
98 RTN
```

Program for HP-67/97

```
001 *LBLe
002 RCL7
003 PSE
004 RCL8
005 RTN
006 *LBLA
007 STO6
008 RCL2
009 ×
010 X⇄Y
011 STO7
012 RCL1
013 ×
014 +
015 RTN
016 *LBLB
017 STO7
018 RCL1
019 ×
020 RCL3
021 X⇄Y
022 -
023 RCL2
024 ÷
025 STO6
026 RTN
027 *LBLC
028 STO8
029 RCL2
030 ×
031 RCL3
032 X⇄Y
033 -
034 RCL1
035 ÷
036 STO7
037 RTN
038 *LBLa
039 STO6
040 RCL5
041 ×
042 X⇄Y
043 STO7
044 RCL4
045 ×
046 +
047 RTN
048 *LBLb
049 STO7
050 RCL4
051 ×
052 RCL6
053 X⇄Y
054 -
055 RCL5
056 ÷
057 STO8
058 RTN
```

```
059 *LBLD
060 STO6
061 RCL5
062 ×
063 RCL6
064 X⇄Y
065 -
066 RCL4
067 ÷
068 STO7
069 RTN
070 *LBLE
071 RCL1
072 RCL6
073 ×
074 RCL4
075 RCL3
076 ×
077 -
078 STO8
079 RCL5
080 RCL3
081 ×
082 RCL2
083 RCL6
084 ×
085 -
086 STO7
087 RCL1
088 RCL5
089 ×
090 RCL4
091 RCL2
092 ×
093 -
094 STO9
095 ÷
096 R/S
097 RCL6
098 RCL9
099 ÷
100 RTN
101 *LBLc
102 RCL7
103 X≠0?
104 GTO0
105 RCL6
106 X≠0?
107 GTO0
108 1
109 RTN
110 *LBL0
111 2
112 RTN
113 *LBLd
114 STO1
115 *LBL1
116 R/S
117 STOi
118 ISZI
119 GTO1
```

Table 9.4

HP-25/25C User Instructions for Part I

Title: Learning about Equations and Their Graphs

Part I: $ax + by = c$

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS				OUTPUT DATA/UNITS
1	Input /store constants	a	GTO	34	R/S		a
		b	R/S				b
		c	R/S				c
2	Execute/iterate 3, 4, 5						
	in any order						
3	Does x,y "fit" equation?	x	ENTER ↑				
	Yes, if c' = c.	y	GTO	05	R/S		c'
4	Given x, what is y?	x	GTO	14	R/S		y
*							
5	Given y, what is x?	y	GTO	24	R/S		x
*							
6	For new equation, go to	1.					
*	To verify, first:		GTO	01	R/S		x **
							y
	Then:		R/S				c
	** Displayed during						
	1-second pause.						

EQUATION:									
x									
y									
c'									

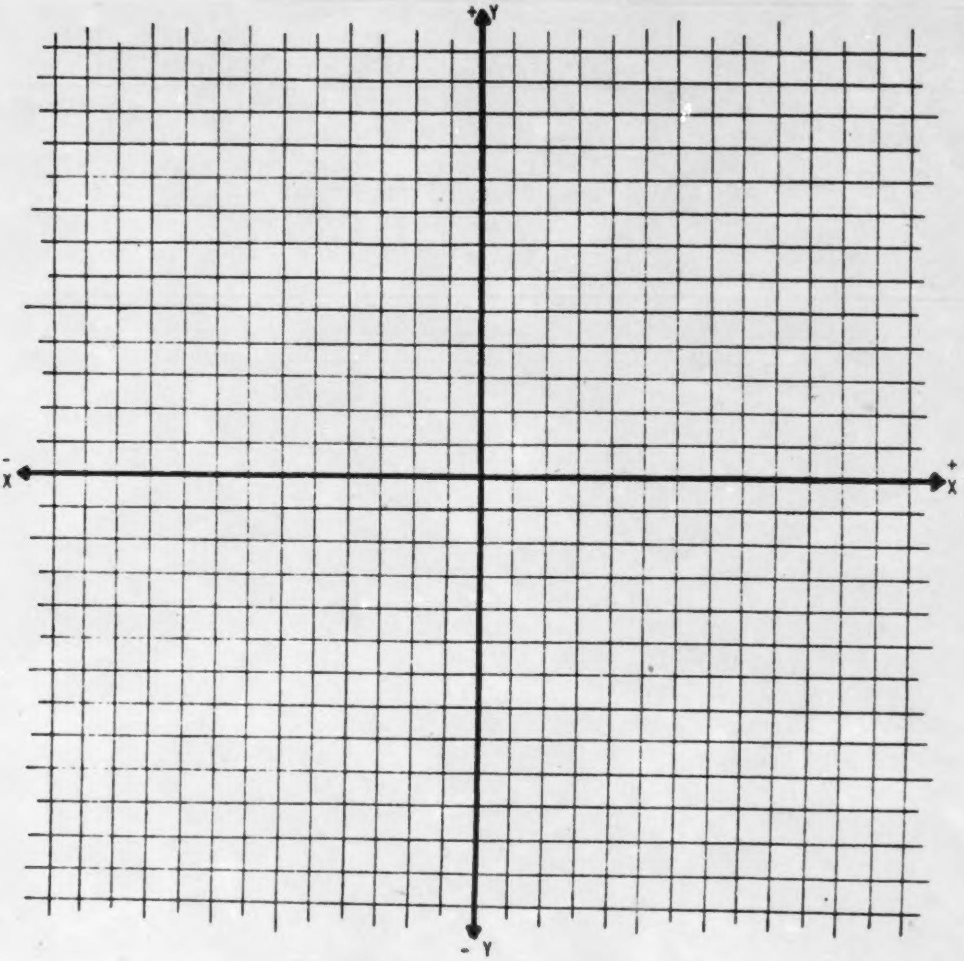
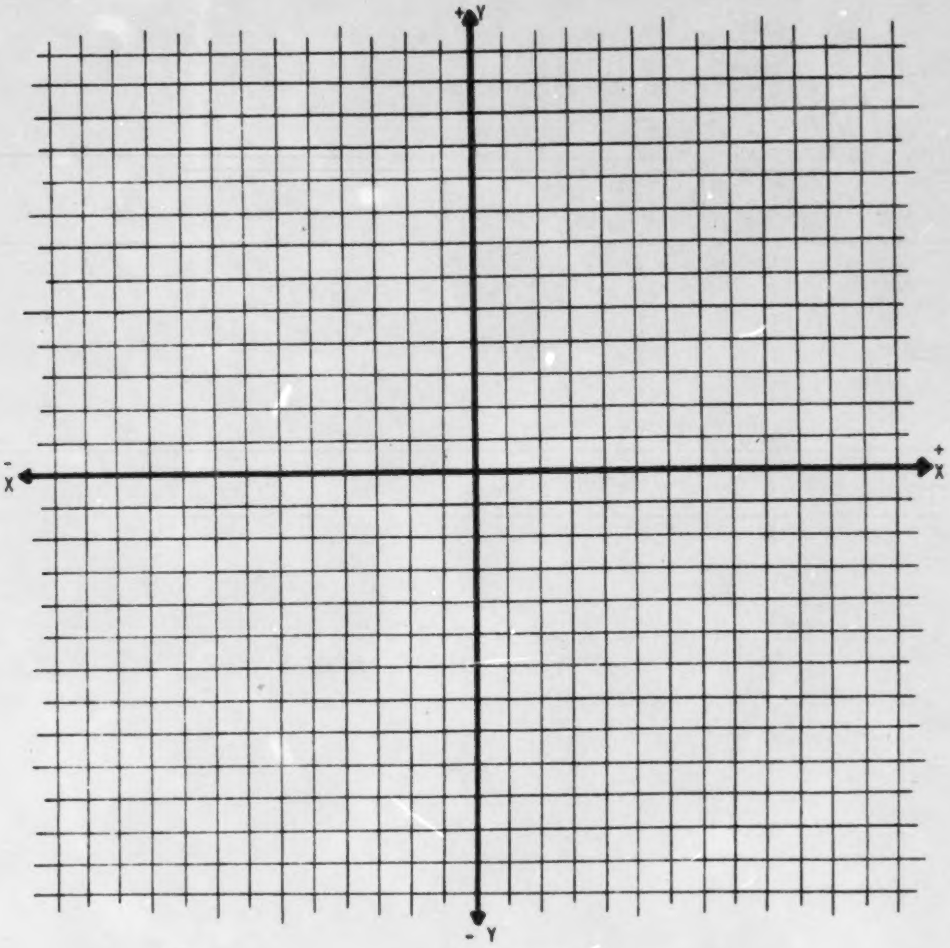


Figure 9.1. Student record sheet for Part I.

EQUATION:									
x									
y									
c'									



EQUATION:									
x									
y									
c"									

Figure 9.2. Student Record Sheet for Part II

Table 9.5

HP-25/25C Program and Register Contents--Part I Only

```
01  RCL7
02   PSE
03  RCL0
04   R/S
05  STO0
06  RCL2
07     x
08   X⇄Y
09  STO7
10  RCL1
11     x
12     +
13   R/S
14  STO7
15  RCL1
16     x
17  RCL3
18   X⇄Y
19     -
20  RCL2
21     ÷
22  STO0
23   R/S
24  STO0
25  RCL2
26     x
27  RCL3
28   X⇄Y
29     -
30  RCL1
31     ÷
32  STO7
33   R/S
34  STO1
35   R/S
36  STO2
37   R/S
38  STO3
39   R/S
```

REGISTERS	
R_1	a
R_2	b
R_3	c
R_4	
R_5	
R_6	
R_7	x
R_0	y

Equations

a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 are rational numbers (or rational approximations to real numbers) such that $a \neq 0$, $b \neq 0$, $a_1 \neq 0$, $b_1 \neq 0$, $a_2 \neq 0$, $b_2 \neq 0$.

Part I

0: $ax + by = c$ 0.1: $y = \dfrac{c - ax}{b}$ 0.2: $x = \dfrac{c - by}{a}$

Part II

1: $a_1x + b_1y = c_1$ 1.1: $y = \dfrac{c_1 - a_1x}{b_1}$ 1.2: $x = \dfrac{c_1 - b_1y}{a_1}$

2: $a_2x + b_2y = c_2$ 2.1: $y = \dfrac{c_2 - a_2x}{b_2}$ 2.2: $x = \dfrac{c_2 - b_2y}{a_2}$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} = \frac{n_x}{d}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{n_y}{d}$$

Note: If $d \neq 0$, a unique solution (x,y) exists for equations 1 and 2.
If $d = 0$ and $n_x = n_y = $), the graphs of equations 1 and 2 are the same line; otherwise, the graphs are distinct parallel lines.

Contents of Storage Registers*

R_1: a or a_1

R_2: b or b_1

R_3: c or c_1

R_4: a_2

R_5: b_2

R_6: c_2

R_7: x or n_x

R_8: y or n_y

R_9: d

Principal Programmer-defined Functions

HP-19C HP-29C	HP-67 HP-97	
1	A	Given a user-selected x and y, calculates c' from equation 0 (or 1); x,y is a root if c' = c (or c_1).
2	B	Given a user-selected x, calculates y from equation 0.1 (or 1.1).
3	C	Given a user-selected y, calculates x from equation 0.2 (or 1.2).
4	a	Given a user-selected x and y, calculates c" from equation 2; x,y is a root if c" = c_2.
5	b	Given a user-selected x, calculates y from equation 2.1.
6	c	Given a user-selected y, calculates x from equation 2.2.
9	E	Calculates common solution (x,y) for a pair of equations 1 and 2.
*	e	If E generates an ERROR display (d = 0), displays 1 or 2 to signify the number of lines associated with the graphs of the pair of equations: only 1 line, or 2 parallel lines.

* In the case of the HP-19C/29C, there was insufficient space to program this function. The determination of 1 vs. 2 lines is effected manually from the keyboard in accord with the User Instructions.

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* For HP-19C,29C,67,97.

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